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On the Nature of Bose-Einstein Condensation in Disordered Systems

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Abstract We study the perfect Bose gas in *random* external potentials and show that there is generalized Bose-Einstein condensation in the random eigenstates if and only if the same occurs in the one-particle kinetic-energy eigenstates, which corresponds to the generalized condensation of the free Bose gas. Moreover, we prove that the amounts of both condensate densities are equal. Our method is based on the derivation of an explicit formula for the occupation measure in the one-body kinetic-energy eigenstates which describes the repartition of particles among these non-random states. This technique can be adapted to re-examine the properties of the perfect Bose gas in the presence of weak (scaled) *non-random* potentials, for which we establish similar results. In addition some of our results can be applied to models with diagonal interactions, that is, models which conserve the occupation density in each single particle eigenstate.

Keywords Generalized Bose-Einstein condensation · Random potentials · Integrated density of states · Lifshitz tails · Diagonal particle interactions

1 Introduction

The study of Bose-Einstein Condensation (BEC) in random media has been an important area for a long time, starting with the papers by Kac and Luttinger, see [1, 2], and then by

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Luttinger and Sy [3]. In the last reference, the authors studied a non-interacting (*perfect*) one dimensional system with point impurities distributed according to the Poisson law, the so-called *Luttinger-Sy model*. The authors conjectured a macroscopic occupation of the random ground state, but this was not rigorously proved until [5]. Although the *free* Bose gas (i.e., the *perfect* gas without external potential) does not exhibit BEC for dimension less than three, the randomness can enhance BEC even in *one dimension*, see e.g. [4]. This striking phenomenon is a consequence of the exponential decay of the one particle density of states at the bottom of the spectrum, known as *Lifshitz tail* , or "doublelogarithmic" asymptotics, which is generally believed to be associated with the existence of localized eigenstates [16].

BEC, however, is usually associated with a macroscopic occupation of the lowest one-particle kinetic-energy eigenstates, which are spatially extended (plane waves). Therefore, it is not immediately clear whether the phenomenon discovered in random boson gases, i.e. macroscopic occupations of localized one-particle states, has any relation to the standard BEC. This is of particular interest in view of the applications of the well-known Bogoliubov c-number approximation [6] to disordered boson systems, see e.g. [12, 13] where the creation/annihilation operators for the kinetic energy ground state are replaced by complex numbers. Although it has been known since the work of Ginibre [7] that this procedure gives the correct pressure in the thermodynamic limit and moreover, it does not require translation invariance, see [8], the associated variational equation (*Condensate Equation*) [9], has a trivial solution unless there is generalized condensate in the lower momentum states. Since such a condensate is not to be expected a priori in random systems, it is therefore interesting to investigate if such type of BEC occurs in some random simple models. One should note that even for translation invariant models, the relation between the solution of the condensate equation and the occupation of the kinetic energy ground state is not straightforward [10].

In this paper, we prove that for the perfect Bose gas in a general class of non-negative random potentials, BEC in the random localized one-particle states and BEC in the lowest one-particle kinetic-energy states occur simultaneously, and moreover the density of the condensate fractions are equal. Our line of reasoning is also applicable to some non-random systems, for example to the case of the perfect gas in weak (scaled) external potentials studied in [24]. We note that our proof for the fact that BEC in the random localized one-particle states *implies* BEC in the lowest one-particle kinetic-energy states holds without modification for a certain class of boson gases with *diagonal interactions* (i.e. invariant with respect to the "local" gauge transformations), while the implication in the other direction requires some additional arguments which will be given in a later work.

The structure of the paper is as follows: in Sect. 2 we describe our disordered system, and in Sect. 3, we recall standard results about the corresponding *perfect* Bose gas. The existence of *generalized* BEC in the eigenstates of the one-particle Schrödinger operator follows from the finite value of the critical density for *any dimension*, which is a consequence of the Lifshitz tail in the limiting Integrated Density of States (IDS). It is well-known that the IDS is a *non-random* quantity, see e.g. [16], and therefore the BEC density is also non-random in the thermodynamic limit. In Sect. 4, we turn to the main result of this paper: we show that this phenomenon occurs *if and only if* there is also occupation of the lowest one-particle kinetic-energy eigenstates. The latter corresponds to the usual generalized BEC in the *free* Bose gas, that is a perfect gas without external potential. To establish this we prove the existence of a non-random limiting occupation measure for kinetic energy eigenstates, and moreover, we obtain an explicit expression for it. To this end, we need some estimates for the IDS before the thermodynamic limit, namely a *finite volume* version of the Lifshitz tail estimates, which we prove in Sect. 5, using techniques developed in [14, 15]. For any *finite* but large enough system, these bounds hold almost surely with respect to random potential realizations. In Sect. 6, we look at the particular case of the Luttinger-Sy model and examine the nature of the condensate in the one-particle kinetic energy eigenstates, showing that although there is generalized BEC, *no condensation* occurs in any of them. In Sect. 7, we describe briefly how the method developed in Sect. 4 applies with minor modifications to a perfect Bose gas in a general class of weak (scaled), non-random external potentials. To make the paper more accessible and easy to read, we postpone some technical estimates concerning random potentials and Brownian motion to Appendices A and B, respectively.

2 Model, Notations and Definitions

Let $\{\Lambda_l := (-l/2, l/2)^d\}_{l\geq 1}$ be a sequence of hypercubes of side l in $\mathbb{R}^d$, $d \geq 1$, centered at the origin of coordinates with volumes $V_l = l^d$. We consider a system of identical bosons, of mass m, contained in Λ_l. For simplicity, we use a system of units such that $\hbar = m = 1$. First we define the self-adjoint one-particle kinetic-energy operator of our system by:

$$h_l^0 := -\frac{1}{2}\Delta_D, \tag{2.1}$$

acting in the Hilbert space $\mathscr{H}_l := L^2(\Lambda_l)$. The subscript D stands for *Dirichlet* boundary conditions. We denote by $\{\psi_k^l, \varepsilon_k^l\}_{k\geq 1}$ the set of normalized eigenfunctions and eigenvalues corresponding to h_l^0. By convention, we order the eigenvalues (counting the multiplicity) as $\varepsilon_1^l \leq \varepsilon_2^l \leq \varepsilon_3^l \leq \cdots$.

We define an external random potential $v^{(\cdot)}(\cdot) : \Omega \times \mathbb{R}^d \to \mathbb{R},\ x \mapsto v^\omega(x)$ as a random field on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, satisfying the following conditions:

(i) $v^\omega, \omega \in \Omega$, is non-negative;
(ii) $p := \mathbb{P}\{\omega : v^\omega(0) = 0\} < 1$.

As usual, we assume that this field is *regular*, *homogeneous* and *ergodic*. These technical conditions are made more explicit in Appendix B. Then the corresponding random Schrödinger operator acting in $\mathscr{H} := L^2(\mathbb{R}^d)$ is a perturbation of the kinetic-energy operator:

$$h^\omega := -\frac{1}{2}\Delta \dotplus v^\omega, \tag{2.2}$$

defined as a sum in the *quadratic-forms* sense. The restriction to the box Λ_l, is specified by the Dirichlet boundary conditions and for regular potentials one gets the self-adjoint operator:

$$h_l^\omega := \left(-\frac{1}{2}\Delta + v^\omega\right)_D = h_l^0 \dotplus v^\omega, \tag{2.3}$$

acting in $\mathscr{H}_l$. We denote by $\{\phi_i^{\omega,l}, E_i^{\omega,l}\}_{i\geq 1}$ the set of normalized eigenfunctions and corresponding eigenvalues of h_l. Again, we order the eigenvalues (counting the multiplicity) so that $E_1^{\omega,l} \leq E_2^{\omega,l} \leq E_3^{\omega,l} \cdots$. Note that the *non-negativity* of the random potential implies that $E_1^{\omega,l} > 0$. So, for convenience we assume also that in the thermodynamic limit *almost surely* (a.s.) with respect to the probability $\mathbb{P}$, the lowest edge of this random one-particle spectrum is:

(iii) a.s.-$\lim_{l\to\infty} E_1^{\omega,l} = 0$.

When no confusion arises, we shall *omit* the explicit mention of l and ω dependence. Note that the non-negativity of the potential implies that:

$$\begin{aligned} &\text{(a)} \quad Q(h_l^\omega) \subset Q(h_l^0), \quad Q \text{ being the quadratic form domain,} \\ &\text{(b)} \quad (\varphi, h_l^\omega \varphi) \geq (\varphi, h_l^0 \varphi), \quad \forall \varphi \in Q(h_l^\omega). \end{aligned} \tag{2.4}$$

Now, we turn to the many-body problem. Let $\mathscr{F}_l := \mathscr{F}_l(\mathscr{H}_l)$ be the symmetric Fock space constructed over $\mathscr{H}_l$. Then $H_l := d\Gamma(h_l^\omega)$ denotes the second quantization of the *one-particle* Schrödinger operator h_l^ω in $\mathscr{F}_l$. Note that the operator H_l acting in $\mathscr{F}_l$ has the form:

$$H_l = \sum_{j \geq 1} E_j^{\omega,l} a^*(\phi_j) a(\phi_j), \tag{2.5}$$

where $a^*(\phi_i), a(\phi_i)$ are the creation and annihilation operators (satisfying the boson *Canonical Commutation Relations*) in the one-particle eigenstates $\{\phi_i := \phi_i^{\omega,l}\}_{i \geq 1}$ of h_l^ω. Then, the grand-canonical Hamiltonian of the perfect Bose gas in a random external potential is given by:

$$H_l(\mu) := H_l - \mu N_l = \sum_{i \geq 1} (E_i^{\omega,l} - \mu) N_l(\phi_i), \tag{2.6}$$

where $N_l(\phi_i) := a^*(\phi_i) a(\phi_i)$ is the operator for the number of particles in the eigenstate ϕ_i, $N_l := \sum_j N_l(\phi_j)$ is the operator for the total number of particles in Λ_l and μ is the chemical potential. Note that N_l can be expanded over *any* basis in the space $\mathscr{H}_l$, and in particular over the one defined by the free one-particle kinetic-energy eigenstates $\{\psi_k^l, \varepsilon_k\}_k$.

Although this paper is mainly devoted to the perfect Bose gas, some of our results can be extended to a class of models with "*diagonal interaction*" in addition to the random potential. By this we mean models with Hamiltonian $H_l^U(\mu) := H_l(\mu) + U_l$, where U_l is a many-body interaction, satisfying the "local" gauge invariance:

$$[H_l^U(\mu), N_l(\phi_j)] = 0 \tag{2.7}$$

for any $j \geq 1$, or equivalently:

$$e^{i\gamma_j N_l(\phi_j)} H_l^U(\mu) e^{-i\gamma_j N_l(\phi_j)} = H_l^U(\mu), \quad \gamma_j \in \mathbb{R}^1, \ j \geq 1. \tag{2.8}$$

The latter means that U_l is a function of the occupation number operators $\{N_l(\phi_j)\}_{j \geq 1}$, and for this reason it is called a "*diagonal interaction*". We shall assume that U_l is bounded from below. A well-known example is the *mean-field* interaction $U_l := \lambda N_l^2 / 2V_l$, $\lambda \geq 0$. [19, 20]. Our results for the general diagonal interaction are weaker than for the mean-field interaction, see Remarks 4.1 and 4.2.

Note that in the free Bose gas, with periodic boundary conditions the "local" gauge invariance (2.7) gives the *same* selection rule as the *momentum conservation law* which ensures that the number of particles in each momentum state is conserved. In the random model there is no such momentum selection rule but in our model it is the particle number in each random eigenstate ϕ_i that is conserved.

We denote by $\langle - \rangle_{H_l^U}$ the equilibrium quantum Gibbs state defined by the Hamiltonian $H_l^U(\mu)$:

$$\langle A \rangle_{H_l^U}(\beta, \mu) := \frac{\mathrm{Tr}_{\mathscr{F}_l}\{\exp(-\beta H_l^U(\mu)) A\}}{\mathrm{Tr}_{\mathscr{F}_l} \exp(-\beta H_l^U(\mu))},$$

and we put $\langle - \rangle_l := \langle - \rangle_{H_l^{U=0}}$. For simplicity, we shall omit in the following the explicit mention of the dependence on the thermodynamic parameters (β, μ). Finally, we define the *Thermodynamic Limit* (TL) as the limit, when $l \to \infty$.

3 Generalized BEC in One-Particle *Random* Eigenstates

In this section we consider the possibility of macroscopic occupation of the one-particle random Schrödinger operator (2.3) eigenstates $\{\phi_i\}_{i\geq 1}$. Recall that the corresponding limiting IDS, $\nu(E)$, is defined as:

$$\nu(E) := \lim_{l\to\infty} \nu_l^{\omega}(E) = \lim_{l\to\infty} \frac{1}{V_l} \sharp\{i : E_i^{\omega,l} \leq E\}. \tag{3.1}$$

Although the finite-volume IDS, $\nu_l^{\omega}(E)$, are random measures, one can check that for homogeneous ergodic random potentials the limit (3.1) has the property of *self-averaging* [16]. This means that $\nu(E)$ is almost surely (a.s.) a *non-random* measure. Let us define a (random) particle density *occupation measures* m_l by:

$$m_l(A) := \frac{1}{V_l} \sum_{i:E_i\in A} \langle N_l(\phi_i)\rangle_l, \quad A \subset \mathbb{R}. \tag{3.2}$$

Then using standard methods, one can prove that this sequence of measures has (a.s.) a non-random weak-limit m, see (3.8) below. Moreover, if the critical density

$$\rho_c := \lim_{\mu\to 0} \int_0^{\infty} \frac{1}{e^{\beta(E-\mu)} - 1} \nu(\mathrm{d}E) \tag{3.3}$$

is finite, then one obtains a *generalized* Bose-Einstein condensation (g-BEC) in the sense that this measure m has an atom at the bottom of the spectrum of the random Schrödinger operator, which by (iii), Sect. 2, is assumed to be at 0:

$$m(\{0\}) = \lim_{\delta\downarrow 0} \lim_{l\to\infty} \sum_{i:E_i\leq\delta} \frac{1}{V_l} \langle N_l(\phi_i)\rangle_l = \begin{cases} 0 & \text{if } \overline{\rho} < \rho_c, \\ \overline{\rho} - \rho_c & \text{if } \overline{\rho} \geq \rho_c, \end{cases} \tag{3.4}$$

where $\overline{\rho}$ denotes a (fixed) mean density [4, 5]. Physically, this corresponds to the macroscopic occupation of the set of eigenstates ϕ_i with energy close to the ground state ϕ_1. However, we have to stress that BEC in this sense does *not* necessarily imply a macroscopic occupation of the ground state. In fact, the condensate can be spread over many (and even infinitely many) states.

These various situations correspond to classification of the g-BEC on the *types* I, II and III, introduced in the eighties by van den Berg-Lewis-Pulé, see e.g. [17] or [6, 18]. The most striking case is type III when generalized BEC occurs in the sense of (3.4) even though *none* of the eigenstates ϕ_i are macroscopically occupied. The realization of different types depends on how the relative gaps between the eigenvalues E_i at the bottom of the spectrum vanishes in the TL. To our knowledge, analysis of this behaviour in random system has only been realised in some particular cases, see [5] for a comprehensive presentation. The concept of *generalized* BEC is more stable then the standard one-mode BEC, since it depends on the global low-energy behaviour of the density of states, especially on its ability to make the critical density (3.3) finite. We note also that, since the IDS (3.1) is *not* random, the same it true for the amount of the g-BEC (3.4).

We can also obtain an explicit expression for the limiting measure m. Note that we have *fixed* the *mean density* $\overline{\rho}$, which implies that we require the chemical potential μ to satisfy the equation:

$$\overline{\rho} = \frac{1}{V_l}\langle N_l\rangle_l(\beta,\mu) = \frac{1}{V_l}\sum_{i\geq 1}\frac{1}{e^{\beta(E_i^{\omega,l}-\mu)}-1}, \tag{3.5}$$

for any l. Since the system is disordered, the unique solution $\mu_l^\omega := \mu_l^\omega(\beta,\overline{\rho})$ of this equation is a *random* variable, which is a.s. non-random in the TL [4, 5]. In the rest of this paper we denote the non-random $\mu_\infty :=$ a.s.-$\lim_{l\to\infty}\mu_l^\omega$. By condition (iii), Sect. 2, and by (3.7) it is a continuous function of $\overline{\rho}$:

$$\mu_\infty(\beta,\overline{\rho}) = \begin{cases} 0 & \text{if } \overline{\rho}\geq\rho_c, \\ \overline{\mu}<0 & \text{if } \overline{\rho}<\rho_c, \end{cases} \tag{3.6}$$

where $\overline{\mu} := \overline{\mu}(\beta,\overline{\rho})$ is a (unique) solution of the equation:

$$\overline{\rho} = \int_0^\infty \frac{1}{e^{\beta(E-\mu)}-1}\nu(\mathrm{d}E), \tag{3.7}$$

for $\overline{\rho}\leq\rho_c$.

Remark 3.1 Note that μ_∞ is non-positive (3.6), which is not true in general for the random finite-volume solution μ_l^ω. Indeed, the only restriction we have is that $\mu_l^\omega < E_1^{\omega,l}$, which is the well-known condition for the pressure of the perfect Bose gas to *exist*. We return to this question in Sect. 4 when we study BEC in the free one-particle kinetic-energy operator eigenfunctions in the presence of a random potential.

We also recall that for (3.6) the explicit expression of the weak limit for the general particle density occupation measure is:

$$m(\mathrm{d}E) = \begin{cases} (\overline{\rho}-\rho_c)\delta_0(\mathrm{d}E) + (e^{\beta E}-1)^{-1}\nu(\mathrm{d}E) & \text{if } \overline{\rho}\geq\rho_c, \\ (e^{\beta(E-\mu_\infty)}-1)^{-1}\nu(\mathrm{d}E) & \text{if } \overline{\rho}<\rho_c. \end{cases} \tag{3.8}$$

We end this section with a comment about the difference between the model of the perfect Bose gas embedded into a random potential and the *free* Bose gas. In the latter case, one should consider the IDS of the one-particle kinetic-energy operator (2.1), which is given by the *Weyl formula*:

$$\nu^0(E) = C_d E^{d/2}, \tag{3.9}$$

where is C_d is a constant term depending only on the dimensionality d. It is known that for this IDS, the critical density (3.3) is finite only when $d>2$, and hence the fact that BEC does not occur for low dimensions. On the other hand, a common feature of Schrödinger operators with regular, stationary, non-negative ergodic random potentials is the so-called *Lifshitz tails* behaviour of the IDS near the bottom of the spectrum. When the lower edge of the spectrum coincides with $E=0$ (condition (iii)), this means roughly that (see for example [16]):

$$\nu(E) \sim e^{-a/E^{d/2}} \tag{3.10}$$

for small E and $a > 0$. Hence, the critical density (3.3) is finite in any dimension, and therefore enhances BEC in the sense of (3.4) even for $d = 1, 2$. This was shown in [4, 5], where some specific examples of one-dimensional *Poisson disordered* systems exhibiting g-BEC in the sense of (3.4) were studied. In this article we require only the following rigorous upper estimate:

$$\lim_{E \to 0^+} (-E^{d/2}) \ln(\nu(E)) \geq a > 0, \tag{3.11}$$

for some constant a. This can be proved (see [14]) under the technical conditions detailed in Appendix B, which are assumed throughout this paper. In particular these conditions are satisfied in the case of Poisson random potentials with sufficiently fast decay of the potential around each impurity.

4 Generalized BEC in One-Particle *Kinetic* Energy Eigenstates

4.1 Occupation Measure for One-Particle Kinetic Energy Eigenstates

Similar to (3.2), we introduce the sequence of particle occupation measure $\tilde{m}_l$ for kinetic energy eigenfunctions $\{\psi_k := \psi_k^l\}_{k \in \Lambda_l^*}$:

$$\tilde{m}_l(A) := \frac{1}{V_l} \sum_{k: \varepsilon_k \in A} \langle N_l(\psi_k) \rangle_l, \quad A \subset \mathbb{R}, \tag{4.1}$$

but now in the *random equilibrium states* $\langle - \rangle_l$ corresponding to the perfect boson gas with Hamiltonian (2.5).

Note that, contrary to the last section, the standard arguments used to prove the existence of a limiting measure in TL are not valid for (4.1), since the kinetic energy operator (2.1) and the random Schrödinger operator (2.3) *do not commute*.

We remark also that even if we know that the measure m (3.8) has an atom at the edge of the spectrum (g-BEC), we cannot deduce that the limiting measure $\tilde{m}$ (assuming that it exists) also manifests g-BEC in the free kinetic energy eigenstates ψ_k.

Now we formulate the main result of this section.

Theorem 4.1 *The sequence of measures $\tilde{m}_l$ converges a.s. in a weak sense to a non-random measure $\tilde{m}$, which is given by*:

$$\tilde{m}(\mathrm{d}\varepsilon) = \begin{cases} (\overline{\rho} - \rho_c)\delta_0(\mathrm{d}\varepsilon) + F(\varepsilon)\mathrm{d}\varepsilon & \text{if } \overline{\rho} \geq \rho_c, \\ F(\varepsilon)\mathrm{d}\varepsilon & \text{if } \overline{\rho} < \rho_c \end{cases}$$

with density $F(\varepsilon)$ defined by:

$$F(\varepsilon) = (2\varepsilon)^{d/2-1} \int_{S_d^1} \mathrm{d}\sigma\, g(\sqrt{2\varepsilon} n_\sigma).$$

Here, S_d^1 denotes the unit sphere in $\mathbb{R}^d$ centered at the origin, n_σ the unit outward drawn normal vector, and $\mathrm{d}\sigma$ the surface measure of S_d^1. The function g is defined as follows

$$g(k) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{d}x\, e^{ikx} \sum_{n \geq 1} e^{n\beta\mu_\infty} \mathbb{E}_\omega\big(K_\omega^{n\beta}(x, 0)\big) \tag{4.2}$$

where $\mathbb{E}_\omega$ *is the expectation on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and* $K^t_\omega(x, x')$ *is the kernel of the operator* e^{-th^ω}.

Note that since the measures $w^{n\beta}$ on $\Omega^{n\beta}_{(0,x)}$ are normalized, we recover from (4.2) the expression for the free Bose gas if we put $v^\omega = 0$.

Before proceeding with the proof, we give some comments about these results.

(a) First, the existence of a non-trivial limiting kinetic energy states occupation measure provides a rigorous basis for discussing the macroscopic occupation of the free Bose gas eigenstates.
(b) Moreover, both occupation measures (3.8) and (4.1) do not only exhibit simultaneously an atom at the bottom of the spectrum, but these atoms have the *same* non-random weights. It is quite surprising that the generalized BEC triggered by the Lifshitz tail in a low dimension disordered system produces the same value of the generalized BEC in the lowest one-particle kinetic-energy states.
(c) In addition our proofs have the following consequence for models with diagonal interaction U_l. The occurrence of generalized BEC in random one-particle states implies there is generalized BEC in the extended, i.e., kinetic-energy eigenstates and the density of the former cannot exceed the density of the latter. Our proof also shows that in spite of the lack of translation invariance in the random system, condensation always occurs in the lower kinetic energy states provided we can prove monotonicity of the finite-volume mean occupation numbers, $\langle N_l(\phi_j)\rangle_{H_l^U}$ as a function of $j \geq 1$, which can be done for the mean-field case.

4.2 Proofs

We start by expanding the measure $\tilde{m}$ in terms of the random equilibrium mean-values of occupation numbers in the corresponding eigenstates ϕ_i. Using the linearity (respectively conjugate linearity) of the creation and annihilation operators one obtains:

$$\begin{aligned}\tilde{m}_l(A) &= \frac{1}{V_l}\sum_{k:\varepsilon_k\in A}\langle a^*(\psi_k)a(\psi_k)\rangle_l \\ &= \frac{1}{V_l}\sum_{i,j}\sum_{k:\varepsilon_k\in A}(\phi_i,\psi_k)\overline{(\phi_j,\psi_k)}\langle a^*(\phi_i)a(\phi_j)\rangle_l \\ &= \frac{1}{V_l}\sum_{i}\sum_{k:\varepsilon_k\in A}|(\phi_i,\psi_k)|^2\langle a^*(\phi_i)a(\phi_i)\rangle_l. \end{aligned} \tag{4.3}$$

In the last equality, we have used the "local" gauge invariance (2.7) which implies that:

$$\langle a^*(\phi_i)a(\phi_j)\rangle_l = 0 \quad \text{if } i \neq j.$$

We first prove two important lemmas.

The first result states that if there is condensation in the lowest random eigenstates $\{\phi_i\}_i$, then there is also condensation in the lowest kinetic-energy states $\{\psi_k\}_k$. Moreover, the amount of the latter condensate density has to be not *less* than the former.

Lemma 4.1 *Let* $\{\tilde{m}_{l_r}\}_{r\geq 1}$ *be a convergent subsequence. We denote by* $\tilde{m}$ *its* (*weak*) *limit. Then*:

$$\tilde{m}(\{0\}) \geq m(\{0\}) = \begin{cases} \overline{\rho} - \rho_c & \text{if } \overline{\rho} \geq \rho_c, \\ 0 & \text{if } \overline{\rho} < \rho_c. \end{cases}$$

Proof Let $\gamma > 0$. Using the expansion of the functions ψ_k in the basis $\{\phi_i\}_{i\geq 1}$, we obtain:

$$\begin{aligned} \tilde{m}([0,\gamma]) &= \lim_{r\to\infty} \frac{1}{V_{l_r}} \sum_{k:\varepsilon_k\leq\gamma} \langle N_{l_r}(\psi_k)\rangle_{l_r} \\ &= \lim_{r\to\infty} \frac{1}{V_{l_r}} \sum_{k:\varepsilon_k\leq\gamma} \sum_{i\geq 1} |(\phi_i,\psi_k)|^2 \langle N_{l_r}(\phi_i)\rangle_{l_r} \\ &\geq \lim_{r\to\infty} \frac{1}{V_{l_r}} \sum_{k:\varepsilon_k\leq\gamma} \sum_{i:E_i\leq\delta} |(\phi_i,\psi_k)|^2 \langle N_{l_r}(\phi_i)\rangle_{l_r} \end{aligned}$$

for any $\delta > 0$. The non-negativity of the random potential (2.4) implies:

$$\begin{aligned} \sum_{k:\varepsilon_k>\gamma} |(\phi_i,\psi_k)|^2 &\leq \sum_{k:\varepsilon_k>\gamma} \frac{\varepsilon_k}{\gamma} |(\phi_i,\psi_k)|^2 \leq \frac{1}{\gamma} \sum_{k\geq 1} \varepsilon_k |(\phi_i,\psi_k)|^2 = \frac{1}{\gamma}(\phi_i, h_l^0 \phi_i) \\ &\leq \frac{1}{\gamma}(\phi_i, h_l^\omega \phi_i) = \frac{E_i^\omega}{\gamma}. \end{aligned}$$

We then obtain:

$$\begin{aligned} \tilde{m}([0,\gamma]) &\geq \lim_{r\to\infty} \frac{1}{V_{l_r}} \sum_{i:E_i\leq\delta} \langle N_{l_r}(\phi_i)\rangle_{l_r} \left(1 - \sum_{k:\varepsilon_k>\gamma} |(\phi_i,\psi_k)|^2\right) \\ &\geq \lim_{r\to\infty} \frac{1}{V_{l_r}} \sum_{i:E_i\leq\delta} \langle N_{l_r}(\phi_i)\rangle_{l_r} (1 - E_i/\gamma) \\ &\geq \lim_{r\to\infty} (1-\delta/\gamma) \frac{1}{V_{l_r}} \sum_{i:E_i\leq\delta} \langle N_{l_r}(\phi_i)\rangle_{l_r} = (1-\delta/\gamma) m([0,\delta]) \geq 0. \end{aligned}$$

But δ is arbitrary, and the lemma follows by letting $\delta \to 0$. □

Remark 4.1 (Diagonal Interaction) The proof of Lemma 4.1 can be readily extended to a version which does not require the sequence of measures $\tilde{m}_l$ to converge. This is valid for models with Hamiltonian H_l^U, which satisfy the invariance condition (2.7) and for which the random potential is non-negative. The equivalent statement is then:

Suppose that the sequence m_l converges to m, then

$$\lim_{\delta\to 0} \liminf_{l\to\infty} \tilde{m}_l([0,\delta]) \geq m(\{0\}).$$

In the next lemma, we show that for the perfect gas the kinetic states occupation measure (4.1) can have an atom in the thermodynamic limit only at zero kinetic energy. We shall not assume that the sequence $\tilde{m}_l$ has a weak limit, instead we consider only some convergent subsequence. Note that at least one such subsequence always exists, see [21], Chap. VIII.6.

Lemma 4.2 *Let* $\{\tilde{m}_{l_r}\}_{r\geq 1}$ *be a convergent subsequence, and* $\tilde{m}$ *be its (weak) limit. Then, it is absolutely continuous on* $\mathbb{R}_+ := (0, \infty)$.

Proof Let A to be a Borel subset of $(0, \infty)$, with Lebesgue measure 0, and let a be such that $\inf A > a > 0$. Then:

$$\begin{aligned}\tilde{m}_{l_r}(A) &= \frac{1}{V_{l_r}} \sum_{k:\varepsilon_k \in A} \langle N_{l_r}(\psi_k)\rangle_{l_r} \\ &= \frac{1}{V_{l_r}} \sum_{k:\varepsilon_k \in A} \sum_i |(\phi_i, \psi_k)|^2 \langle N_{l_r}(\phi_i)\rangle_{l_r} \\ &= \frac{1}{V_{l_r}} \sum_{k:\varepsilon_k \in A} \sum_{i:E_i \leq \alpha} |(\phi_i, \psi_k)|^2 \langle N_{l_r}(\phi_i)\rangle_{l_r} \\ &\quad + \frac{1}{V_{l_r}} \sum_{k:\varepsilon_k \in A} \sum_{i:E_i > \alpha} |(\phi_i, \psi_k)|^2 \langle N_{l_r}(\phi_i)\rangle_{l_r} \end{aligned} \tag{4.4}$$

for some $\alpha > 0$. Next, we use (2.4) to get the following estimate:

$$E_i^\omega = (\phi_i, h_l^\omega \phi_i) \geq (\phi_i, h_l^0 \phi_i) = \sum_k \varepsilon_k |(\phi_i, \psi_k)|^2 \geq a \sum_{k:\varepsilon_k \in A} |(\phi_i, \psi_k)|^2.$$

Since the equilibrium value of the occupation numbers $\langle N_l(\phi_i)\rangle_l = \{e^{E_i^\omega - \mu} - 1\}^{-1}$ are decreasing with i, the estimate (4.4) implies:

$$\tilde{m}_{l_r}(A) \leq \frac{1}{V_{l_r}} \frac{1}{a} \sum_{i:E_i \leq \alpha} E_i^\omega \langle N_{l_r}(\phi_i)\rangle_{l_r} + \langle N_{l_r}(\phi_{i_\alpha})\rangle_{l_r} \frac{1}{V_{l_r}} \sum_{k:\varepsilon_k \in A} 1, \tag{4.5}$$

where ϕ_{i_α} denotes the eigenstate of h_l^ω with the *smallest* eigenvalue *greater* than α. Using again the monotonicity and the finite-volume IDS (3.1) we can get an upper bound for the mean occupation number in the second term of (4.5), since:

$$\overline{\rho} = \frac{1}{V_l} \sum_i \langle N_l(\phi_i)\rangle_l \geq \frac{1}{V_l} \sum_{i:E_i \leq \alpha} \langle N_l(\phi_i)\rangle_l \geq \langle N_l(\phi_{i_\alpha})\rangle_l \nu_l^\omega(\alpha). \tag{4.6}$$

Combining (4.5) and (4.6) we obtain:

$$\tilde{m}_{l_r}(A) \leq \frac{\alpha \overline{\rho}}{a} + \frac{\overline{\rho}}{\nu_{l_r}^\omega(\alpha)} \int_A \nu_{l_r}^0(\mathrm{d}\varepsilon). \tag{4.7}$$

Since the measure ν^0 (3.9) is absolutely continuous with respect to the Lebesgue measure, and $\nu(\alpha)$ is strictly positive for any $\alpha > 0$ the limit $r \to \infty$ in (4.7) gives:

$$\tilde{m}(A) \leq \frac{\alpha \overline{\rho}}{a},$$

but $\alpha > 0$ can be chosen arbitrary small and thus $\tilde{m}(A) = 0$. To finish the proof, note that any Borel subset of $(0, \infty)$ can be expressed as a countable union of disjoint subsets with non-zero infimum. Our arguments than can be applied to each of them. □

Remark 4.2 (Diagonal Interaction) Lemma 4.2 can also be extended in the same way as proposed in Remark 4.1, for Lemma 4.1. Again we assume the invariance condition (2.7) for interacting bosons with Hamiltonian H_l^U and the non-negativity of the random potential, with the additional requirement that the occupation numbers $\langle N_l(\phi_i)\rangle_{H_l^U}$ are monotonic in i. This last property is valid for the Bose-gas with a mean-field interaction, see [11].

Above we exploited the fact that the sequence $\{\tilde{m}_l\}_{l\geq 1}$ has at least one accumulation point. However, to prove convergence, we need to make use of some particular and explicit features of the perfect Bose gas, as well as more detailed information about the properties of the external (random) potential. In particular, we shall need some estimates of the (random) finite volume integrated density of states, see Lemma 5.1.

To this end let us denote by P_A the orthogonal projection onto the subspace spanned by the one-particle kinetic energy states ψ_k with kinetic energy $\varepsilon(k)$ in the set A. Then using the explicit expression for the mean occupation $\langle a^*(\phi_i)a(\phi_i)\rangle_l$ and (4.3) we obtain:

$$\tilde{m}_l(A) = \frac{1}{V_l}\operatorname{Tr} P_A(e^{\beta(h_l^\omega-\mu_l)}-1)^{-1} = \sum_{n\geq 1}\frac{1}{V_l}\operatorname{Tr} P_A(e^{-n\beta(h_l^\omega-\mu_l)}). \tag{4.8}$$

Now we split the measure (4.8) into two parts:

$$\begin{aligned}
&\tilde{m}_l = \tilde{m}_l^{(1)} + \tilde{m}_l^{(2)},\\
&\quad\tilde{m}_l^{(1)}(A) := \sum_{n\geq 1}\frac{1}{V_l}\operatorname{Tr} P_A(e^{-n\beta(h_l^\omega-\mu_l)})\mathbf{1}(\mu_l\leq 1/n),\\
&\quad\tilde{m}_l^{(2)}(A) := \sum_{n\geq 1}\frac{1}{V_l}\operatorname{Tr} P_A(e^{-n\beta(h_l^\omega-\mu_l)})\mathbf{1}(\mu_l > 1/n).
\end{aligned} \tag{4.9}$$

Note that since the chemical potential satisfies (3.5), $\mu_l := \mu_l^\omega$, the indicator functions $\mathbf{1}(\mu_l\leq 1/n)$ and $\mathbf{1}(\mu_l > 1/n)$ split the range of n into the sums (4.9) in a random and volume-dependent way.

We start with the proof of existence of a weak limit of the sequence of random measures $\tilde{m}_l^{(1)}$:

Theorem 4.2 *Let random potential v^ω satisfy the assumptions* (i)–(iii) *of Sect.* 2. *Then for any $d\geq 1$, the sequence of Laplace transforms of the measures $\tilde{m}_l^{(1)}$*:

$$f_l(t;\beta,\mu_l) := \int_{\mathbb{R}} \tilde{m}_l^{(1)}(\mathrm{d}\varepsilon)\, e^{-t\varepsilon} \tag{4.10}$$

converges for any $t>0$ to a (*non-random*) *limit $f(t;\beta,\mu_\infty)$, which is given by*:

$$f(t;\beta,\mu_\infty) = \sum_{n\geq 1} e^{n\beta\mu_\infty}\int_{\mathbb{R}^d}\mathrm{d}x\,\frac{e^{-\|x\|^2/2t}}{(4\pi^2 t)^{d/2}}\mathbb{E}_\omega\big(K_\omega^{n\beta}(x,0)\big). \tag{4.11}$$

Here $\mathbb{E}_\omega$ denotes the expectation with respect to realizations (*configurations*) *ω of the random potential. Note that the sum on the right-hand side converges for all* (*non-random*) *$\mu_\infty\leq 0$, including* 0, *which corresponds to the case $\overline{\rho}\geq\rho_c$.*

Proof By definition of P_A the Laplace transformation (4.10) can be written as:

$$f_l(t;\beta,\mu_l) = \sum_{n\geq 1} \frac{1}{V_l} \operatorname{Tr} e^{-th_l^0}(e^{-n\beta(h_l^\omega-\mu_l)})\mathbf{1}(\mu_l \leq 1/n). \tag{4.12}$$

Now we have to show the uniform convergence of the sum over n to be able to take the term by term limit with respect to l. Since for any bounded operator A and for any trace-class non-negative operator B one has $\operatorname{Tr} AB \leq \|A\| \operatorname{Tr} B$, we get

$$\begin{aligned} a_l(n) &:= \frac{1}{V_l} \operatorname{Tr} e^{-th_l^0} e^{-n\beta(h_l^\omega-\mu_l)}\mathbf{1}(\mu_l \leq 1/n) \\ &\leq \frac{1}{V_l} \operatorname{Tr} e^{-n\beta(h_l^\omega-\mu_l)}\mathbf{1}(\mu_l \leq 1/n). \end{aligned} \tag{4.13}$$

For $\overline{\rho} < \rho_c$, the uniform convergence in (4.11) is immediate. Indeed, for l large enough, the chemical potential satisfies $\mu_l < \mu_\infty/2 < 0$, which by (3.1) provides the following a.s. estimate for (4.13):

$$a_l(n) \leq e^{n\beta\mu_\infty/2} \int_{[0,\infty)} \nu_l^\omega(\mathrm{d}E) e^{-\beta E} \leq K_1 e^{n\beta\mu_\infty/2}, \tag{4.14}$$

with some constant K_1.

However, for the case $\overline{\rho} \geq \rho_c$, this approach does not work, since, in fact, for any finite l the solutions $\mu_l = \mu_l^\omega$ of (3.5) could be *positive* with some probability, event though by condition (iii) (see Sect. 2) it has to *vanish* a.s. in the TL. We use, therefore, the bound:

$$\begin{aligned} a_l(n) &\leq a_l^1(n) + a_l^2(n), \\ a_l^1(n) &:= \frac{1}{V_l} e^\beta \sum_{\{i: E_i^{\omega,l} \leq 1/n^{1-\eta}\}} e^{-n\beta E_i^{\omega,l}}, \\ a_l^2(n) &:= \frac{1}{V_l} e^\beta \sum_{\{i: E_i^{\omega,l} > 1/n^{1-\eta}\}} e^{-n\beta E_i^{\omega,l}}, \end{aligned}$$

which follows, for some $0 < \eta < 1$, from the constraint $\mu_l n \leq 1$ due to the indicator function in (4.13). Then the first term is bounded from above by:

$$a_l^1(n) \leq e^\beta \nu_l^\omega(n^{\eta-1}).$$

On the other hand, by Theorem 5.1 (*finite-volume* Lifshitz tails), for $\alpha > 0$ and $0 < \gamma < d/2$, there exists a subset $\widetilde{\Omega} \subset \Omega$ of full measure, $\mathbb{P}(\widetilde{\Omega}) = 1$, such that for any $\omega \in \widetilde{\Omega}$ there exists a positive finite energy $\mathcal{E}(\omega) := \mathcal{E}_{\alpha,\gamma}(\omega) > 0$ for which one obtains:

$$\nu_l^\omega(E) \leq e^{-\alpha/E^\gamma},$$

for all $E < \mathcal{E}(\omega)$. Therefore, for any configuration $\omega \in \widetilde{\Omega}$ (i.e. almost surely) we have the *volume independent* estimate for all $n > \mathcal{N}(\omega) := \mathcal{E}(\omega)^{1/(\eta-1)}$:

$$a_l^1(n) \leq e^\beta e^{-\alpha n^{(1-\eta)\gamma}}. \tag{4.15}$$

To estimate the coefficients $a_l^2(n)$ from above, we use the upper bound:

$$a_l^2(n) \leq \int_{[1/n^{1-\eta},\infty)} \nu_l^{\omega}(\mathrm{d}E)\, e^{-n\beta E} \leq e^{-\beta n^{\eta}/2} \int_{[1/n^{1-\eta},\infty)} \nu_l^{\omega}(\mathrm{d}E)\, e^{-n\beta E/2}$$
$$\leq e^{-\beta n^{\eta}/2} \int_{[0,\infty)} \nu_l^{\omega}(\mathrm{d}E)\, e^{-\beta E/2}.$$

Then for some $K_2 > 0$ independent of l we obtain:

$$a_l^2(n) \leq K_2 e^{-\beta n^{\eta}/2}. \tag{4.16}$$

Therefore, by (4.14) in the case $\overline{\rho} < \rho_c$, and by (4.15), (4.16) for $\overline{\rho} \geq \rho_c$, we find that there exists a sequence $a(n)$ (independent of l) such that:

$$a_l(n) \leq a(n) \quad \text{and} \quad \sum_{n\geq 1} a(n) < \infty. \tag{4.17}$$

Thus, the series (4.12) is uniformly convergent in l, and one can exchange sum and the limit:

$$\lim_{l\to\infty} f_l(t) = \lim_{l\to\infty} \sum_{n=0}^{\infty} a_l(n) = \sum_{n=0}^{\infty} \lim_{l\to\infty} a_l(n).$$

The rest of the proof is largely inspired by the paper [4]. Let

$$\Omega^T_{(x,x')} := \{\xi : \xi(0) = x, \xi(T) = x'\}$$

be the set of continuous trajectories (paths) $\{\xi(s)\}_{s=0}^{T}$ in $\mathbb{R}^d$, connecting the points x, x', and let w^T denote the normalized Wiener measure on this set. Using the Feynman-Kac representation, we obtain the following limit:

$$\lim_{l\to\infty} a_l(n) = \lim_{l\to\infty} \frac{1}{V_l} \operatorname{Tr} e^{-th_l^0} e^{-n\beta(h_l^{\omega}-\mu_l)} \mathbf{1}(\mu_l \leq 1/n)$$
$$= \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \int_{\Lambda_l} \mathrm{d}x \mathrm{d}x' e^{-th_l^0}(x,x') e^{-n\beta(h_l^{\omega}-\mu_l)}(x',x)$$
$$= e^{n\beta\mu_\infty} \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \int_{\Lambda_l} \mathrm{d}x \mathrm{d}x' \frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}}$$
$$\times \int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi) e^{-\int_0^{n\beta} \mathrm{d}s v^{\omega}(\xi(s))} \chi_{\Lambda_l,n\beta}(\xi) \int_{\Omega^t_{(x,x')}} w^t(\mathrm{d}\xi') \chi_{\Lambda_l,t}(\xi'), \tag{4.18}$$

where we denote by $\chi_{\Lambda_l,T}(\xi)$ the characteristic function of paths ξ such that $\xi(t) \in \Lambda_l$ for all $0 < t < T$. Using Lemma A.2, we can eliminate these restrictions, and also extend one spatial integration over the whole space:

$$\lim_{l\to\infty} \frac{1}{V_l} \operatorname{Tr} e^{-th_l^0} e^{-n\beta(h_l^{\omega}-\mu_l)}$$
$$= e^{n\beta\mu_\infty} \lim_{l\to\infty} \int_{\mathbb{R}^d} \mathrm{d}x \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x' \frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi) e^{-\int_0^{n\beta} \mathrm{d}s v^{\omega}(\xi(s))}. \tag{4.19}$$

Now, by the *ergodic* theorem, we obtain:

$$
\begin{aligned}
&\lim_{l\to\infty} a_l(n)\\
&= \lim_{l\to\infty} \frac{1}{V_l} \operatorname{Tr} e^{-th_l^0} e^{-n\beta(h_l^\omega-\mu_l)}\\
&= e^{n\beta\mu_\infty} \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x' \left\{ \int_{\mathbb{R}^d} \mathrm{d}x \frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi) e^{-\int_0^{n\beta} \mathrm{d}s\, v^\omega(\xi(s))} \right\}\\
&= e^{n\beta\mu_\infty} \mathbb{E}_\omega \left\{ \int_{\mathbb{R}^d} \mathrm{d}x \frac{e^{-\|x\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \int_{\Omega^{n\beta}_{(0,x)}} w^{n\beta}(\mathrm{d}\xi) e^{-\int_0^{n\beta} \mathrm{d}s\, v^\omega(\xi(s))} \right\}.
\end{aligned}
\tag{4.20}
$$

We then get the explicit expression for the limiting Laplace transform:

$$
f(t;\beta,\mu_\infty) = \sum_{n\geq 1} e^{n\beta\mu_\infty} \int_{\mathbb{R}^d} \mathrm{d}x \frac{e^{-\|x\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \mathbb{E}_\omega \left\{ \int_{\Omega^{n\beta}_{(0,x)}} w^{n\beta}(\mathrm{d}\xi) e^{-\int_0^{n\beta} \mathrm{d}s\, v^\omega(\xi(s))} \right\},
$$

which finishes the proof. □

Corollary 4.1 *For any $\overline{\rho}$ the sequence of random measures $\tilde{m}_l^{(1)}$ converges a.s. in the weak sense to a bounded, absolutely continuous non-random measure $\tilde{m}^{(1)}$, with density $F(\varepsilon)$ given by*

$$
F(\varepsilon) := (2\varepsilon)^{d/2-1} \int_{S_d^1} \mathrm{d}\sigma\, g(\sqrt{2\varepsilon} n_\sigma).
$$

Here, S_d^1 denotes the unit sphere in $\mathbb{R}^d$, n_σ the outward drawn normal unit vector, $\mathrm{d}\sigma$ the surface measure on S_d^1 and the function g has the form

$$
g(k) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{d}x e^{ikx} \sum_{n\geq 1} e^{n\beta\mu_\infty} \mathbb{E}_\omega \big(K_\omega^{n\beta}(x,0) \big). \tag{4.21}
$$

Proof By Theorem 4.2, the existence of the weak limit $\tilde{m}^{(1)}$ follows from the existence of the limiting Laplace transform. Moreover, we have the following explicit expression:

$$
\begin{aligned}
&\int_{\mathbb{R}} \tilde{m}^{(1)}(\mathrm{d}\varepsilon) e^{-t\varepsilon}\\
&= \int_{\mathbb{R}^d} \mathrm{d}x \frac{e^{-\|x\|^2/2t}}{(2\pi t)^{d/2}} \sum_{n\geq 1} e^{n\beta\mu} \frac{e^{-\|x\|^2/2n\beta}}{(2\pi n\beta)^{d/2}} \mathbb{E}_\omega \left\{ \int_{\Omega^{n\beta}_{(0,x)}} w^{n\beta}(\mathrm{d}\xi) e^{-\int_0^{n\beta} \mathrm{d}s\, v^\omega(\xi(s))} \right\}\\
&= \int_{[0,\infty)} \mathrm{d}r e^{-t\|r\|^2/2} r^{d-1} \int_{S_d^1} \mathrm{d}\sigma\, g(rn_\sigma)\\
&= \int_{[0,\infty)} \mathrm{d}\varepsilon e^{-t\varepsilon} (2\varepsilon)^{d/2-1} \int_{S_d^1} \mathrm{d}\sigma\, g(\sqrt{2\varepsilon} n_\sigma),
\end{aligned}
$$

which proves the corollary. □

Corollary 4.2 *The measure* $\tilde{m}^{(1)}$ *satisfies the following property*:

$$\int_{[0,\infty)} \tilde{m}^{(1)}(\mathrm{d}\varepsilon) = \begin{cases} \overline{\rho} & \text{if } \overline{\rho} < \rho_c, \\ \rho_c & \text{if } \overline{\rho} \geq \rho_c. \end{cases}$$

Proof By virtue of (4.12) we have:

$$\int_{[0,\infty)} \tilde{m}^{(1)}(\mathrm{d}\varepsilon) = f(0;\beta,\mu_\infty) = \lim_{l\to\infty} \sum_{n\geq 1} \frac{1}{V_l} \operatorname{Tr} e^{-n\beta(h_l^\omega - \mu_l)} \mathbf{1}(\mu_l \leq 1/n).$$

Note that by uniformity of convergence of the sum, see (4.15), (4.16), we can take the limit term by term (for any value of $\overline{\rho}$), and then:

$$\begin{aligned} \int_{[0,\infty)} \tilde{m}^{(1)}(\mathrm{d}\varepsilon) &= \sum_{n\geq 1} \lim_{l\to\infty} \frac{1}{V_l} \operatorname{Tr} e^{-n\beta(h_l^\omega - \mu_l)} \\ &= \sum_{n\geq 1} \int_{[0,\infty)} \nu(\mathrm{d}E) e^{-n\beta(E-\mu_\infty)} \\ &= \int_{[0,\infty)} \nu(\mathrm{d}E)(e^{\beta(E-\mu_\infty)} - 1)^{-1}, \end{aligned}$$

where we use Fubini's theorem for the last step. □

We are now ready for the proof of the main result of this section:

Proof of Theorem 4.1 We first treat the case $\overline{\rho} < \rho_c$. In this situation, the measure $\tilde{m}_l^{(2)}$ is equal to 0 for l large enough, see (4.9), since the solution $\lim_{l\to\infty} \mu_l^\omega$ (3.5) in the TL is a.s. strictly negative. Thus, the total occupation measure $\tilde{m}_l$ is reduced to $\tilde{m}_l^{(1)}$ and the theorem follows from Corollary 4.1.

Now, consider the case $\overline{\rho} \geq \rho_c$. Choose a subsequence l_r such that the total kinetic-energy states occupation measures $\tilde{m}_{l_r}$ converge weakly and a.s., and let the measure $\tilde{m}$ be its limit. By Corollary 4.1, all subsequences of measures $\tilde{m}_{l_r}^{(1)}$ converge to the limiting measure $\tilde{m}^{(1)}$. Therefore, by (4.9), we obtain the weak a.s. convergence:

$$\lim_{r\to\infty} \tilde{m}_{l_r}^{(2)} =: \tilde{m}^{(2)}.$$

By Lemma 4.2, we know that the measure $\tilde{m}$ is absolutely continuous on $(0,\infty)$, and by Corollary 4.1 that $\tilde{m}^{(1)}$ is absolutely continuous on $[0,\infty)$. Therefore we get:

$$\tilde{m}^{\text{a.c.}} = \tilde{m}^{(1)} + \tilde{m}^{(2)\text{a.c.}},$$

where a.c. denotes the *absolute continuous* components.

By definition of the total measure (4.9), $\tilde{m}([0,\infty)) = \overline{\rho}$ and by Lemma 4.1, $\tilde{m}(\{0\}) \geq \overline{\rho} - \rho_c$. Thus, $\tilde{m}((0,\infty)) \leq \rho_c$ and by Corollary 4.2, we can then deduce that the measure $\tilde{m}^{(2)}$ has no absolutely continuous component and therefore consists at most of an atom at $\varepsilon = 0$. Consequently, the full measure $\tilde{m}$ can be expressed as:

$$\tilde{m} = \tilde{m}^{\text{a.c.}} + b\delta_0 = \tilde{m}^{(1)} + b\delta_0,$$

and since by Corollary 4.2

$$b = \overline{\rho} - \int_{\mathbb{R}_+} \tilde{m}^{\text{a.c.}}_{l_r}(\mathrm{d}\varepsilon) = \overline{\rho} - \int_{\mathbb{R}_+} \tilde{m}^{(1)}_{l_r}(\mathrm{d}\varepsilon) = \overline{\rho} - \rho_c$$

for the converging subsequence $\tilde{m}_{l_r}$, we have:

$$\lim_{l_r \to \infty} \tilde{m}_{l_r} = \tilde{m}^{(1)} + (\overline{\rho} - \rho_c)\delta_0.$$

By (4.22) and Corollary 4.1, this limit is independent of the subsequence. Then, the limit of any convergent subsequence is the same, and therefore, using Feller's selection theorem, see [21], Chap. VIII.6, the total kinetic states occupation measures $\tilde{m}_l$ converge weakly to this limit. □

5 Finite Volume Lifshitz Tails

In this section, we give the proof of one important building block of our analysis, Theorem 5.1 about the *finite-volume* Lifshitz tails. Recall that this behaviour is a well-known feature of disordered systems, essentially meaning that for Shrödinger operators which are semi-bounded from below, there are exponentially few eigenstates with energy close to the bottom of the spectrum. To our knowledge, however, this is always shown only in the *infinite-volume* limit, see e.g. [16]. Here, we derive a *finite-volume* estimate for the density of states, uniformly in l, though it could be trivial for small volumes. As one would expect our result is weaker than the asymptotic one, in the sense that we prove it for Lifshitz exponent smaller than the limiting one.

Theorem 5.1 *Let the random potential v^ω satisfy the assumptions* (i)–(iii) *of Sect.* 2. *Then for any $\alpha > 0$ and $0 < \gamma < d/2$, there exists a set $\widetilde{\Omega} \subset \Omega$ of full measure, $\mathbb{P}(\widetilde{\Omega}) = 1$, such that for any configuration $\omega \in \widetilde{\Omega}$ one can find a positive finite energy $\mathcal{E}(\omega) := \mathcal{E}_{\alpha,\gamma}(\omega)$, for which one has the estimate*:

$$\nu_l^\omega(E) \le e^{-\alpha/E^\gamma}$$

for all $E < \mathcal{E}(\omega)$ and for all l.

Remark 5.1 We want to stress that the statement in Theorem 5.1 is valid for all l, but of course, it can be trivial for small l. For example from the positivity of the potential we know that $\nu_l^\omega(E) = 0$ for $E < \pi^2 d/l^2$ and therefore the estimate is trivial for $l < \pi/\sqrt{\mathcal{E}(\omega)}$.

For the proof, we first need a result from [14].

Lemma 5.1 *By assumption* (ii) (*Sect.* 2) *one has,*

$$p = \mathbb{P}\big\{\omega : v^\omega(0) = 0\big\} < 1.$$

Let $\alpha > p/(1-p)$, $B = \pi/(1+\alpha)$, and $E_1^{\omega,l,N} := E_1^{\omega,N}$ be the first eigenvalue of the random Schrödinger operator (2.3) *with Neumann* (*instead of Dirichlet*) *boundary conditions. Then, for l large enough, there exists an independent of l constant $A = A(\alpha)$, such that*

$$\mathbb{P}\big\{\omega : E_1^{\omega,N} < B/l^2\big\} < e^{-AV_l}. \tag{5.1}$$

Detailed conditions on the random potential and a sketch of the proof of this lemma are given in Appendix B. Now we use Lemma 5.1 to prove the following result:

Lemma 5.2 *Assume that the random potential satisfies the assumptions of Lemma 5.1. Then for any $\alpha > 0$ and $0 < \gamma < d/2$,*

$$\sum_{n\geq 1} \mathbb{P}\big\{\sharp\big\{i : E_i^{\omega,l} < 1/n\big\} > V_l e^{-\alpha n^\gamma}, \textit{ for some } l \geq 1\big\} < \infty.$$

Proof Notice that

$$\sum_{n\geq 1} \mathbb{P}\big\{\sharp\big\{i : E_i^{\omega,l} < 1/n\big\} > V_l e^{-\alpha n^\gamma}, \text{ for some } l \geq 1\big\} = \sum_{n\geq 1} \mathbb{P}\Big\{\bigcup_{l\geq 1} S_l^n\Big\}, \tag{5.2}$$

where S_l^n is the set

$$S_l^n := \left\{\omega : \sharp\left\{i : E_i^{\omega,l} < \frac{1}{n}\right\} > V_l e^{-\alpha n^\gamma}\right\}.$$

The sum in the right-hand side of (5.2) does not provide a very useful upper bound, since the sets S_l^n are highly overlapping. We thus need to define a new refined family of sets to avoid this difficulty.

To this end we let $[a]_+$ be the smallest integer $\geq a$, and we define the family of sets:

$$V_k^n := \left\{\omega : \sharp\left\{i : E_i^{\omega,[(ke^{\alpha n^\gamma})^{1/d}]_+} < \frac{1}{n}\right\} \geq k\right\}.$$

Let $k := [V_l e^{-\alpha n^\gamma}]_+$. Since $V_l = l^d$, this implies that $h_l^\omega \geq h^\omega_{[(ke^{\alpha n^\gamma})^{1/d}]_+}$, and therefore:

$$\sharp\left\{i : E_i^{\omega,[(ke^{\alpha n^\gamma})^{1/d}]_+} < \frac{1}{n}\right\} \geq \sharp\left\{i : E_i^{\omega,l} < \frac{1}{n}\right\}.$$

If now $\omega \in S_l^n$, then by the definition of k we obtain:

$$\sharp\left\{i : E_i^{\omega,l} < \frac{1}{n}\right\} \geq k,$$

since the left-hand side is itself an integer. Thus, $S_l^n \subset V_k^n$ and:

$$\mathbb{P}\Big(\bigcup_{l\geq 1} S_l^n\Big) \leq \mathbb{P}\Big(\bigcup_{k\geq 1} V_k^n\Big). \tag{5.3}$$

We define also the sets:

$$W_k^n := \left\{\omega : \sharp\left\{i : E_i^{\omega,[(ke^{\alpha n^\gamma})^{1/d}]_+} < \frac{1}{n}\right\} = k\right\}. \tag{5.4}$$

Let $\omega \in (V_k^n \setminus W_k^n)$. Then by $h^\omega_{[((k+1)e^{\alpha n^\gamma})^{1/d}]_+} \leq h^\omega_{[(ke^{\alpha n^\gamma})^{1/d}]_+}$ we get:

$$\sharp\left\{i : E_i^{\omega,[((k+1)e^{\alpha n^\gamma})^{1/d}]_+} < \frac{1}{n}\right\} \geq \sharp\left\{i : E_i^{\omega,[(ke^{\alpha n^\gamma})^{1/d}]_+} < \frac{1}{n}\right\} \geq k+1.$$

Hence, $(V_k^n \setminus W_k^n) \subset V_{k+1}^n$, and therefore we have for any fixed n and k:

$$V_k^n \subset W_k^n \cup V_{k+1}^n. \tag{5.5}$$

Applying this inclusion M times, for $k = 1, \dots, M$, we obtain:

$$\bigcup_{k=1}^{M} V_k^n \subset \left(W_1^n \cup \bigcup_{k=2}^{M} V_k^n \right) \subset \left(W_1^n \cup W_2^n \cup \bigcup_{k=2}^{M} V_k^n \right) \subset \cdots \subset \left(\bigcup_{k=1}^{M} W_k^n \right) \cup V_{M+1}^n. \tag{5.6}$$

Then we take the limit $M \to \infty$ to recover the infinite union that one needs in (5.3) and we use the inclusion (5.6) to find the inequality:

$$\begin{aligned} \mathbb{P}\left(\bigcup_{k\geq 1} V_k^n \right) &= \lim_{M\to\infty} \mathbb{P}\left(\bigcup_{k=1}^{M} V_k^n \right) \\ &\leq \lim_{M\to\infty} \left(\sum_{k=1}^{M} \mathbb{P}\big(W_k^n\big) + \mathbb{P}\big(V_{(M+1)}^n\big) \right) = \sum_{k=1}^{\infty} \mathbb{P}\big(W_k^n\big) + \lim_{M\to\infty} \mathbb{P}\big(V_M^n\big). \end{aligned} \tag{5.7}$$

The limit in the last term can be calculated directly:

$$\begin{aligned} \lim_{M\to\infty} \mathbb{P}(V_M^n) &= \lim_{M\to\infty} \mathbb{P}\left\{ \omega : \sharp\left\{ i : E_i^{\omega,[(Me^{\alpha n^\gamma})^{1/d}]_+} < \frac{1}{n} \right\} \geq M \right\} \\ &= \lim_{M\to\infty} \mathbb{P}\left\{ \omega : \nu^{\omega}_{[(Me^{\alpha n^\gamma})^{1/d}]_+}(1/n) \geq \frac{M}{[(Me^{\alpha n^\gamma})^{1/d}]_+^d} \right\} \\ &= \mathbb{P}\big\{ \omega : \nu(1/n) \geq K e^{-\alpha n^\gamma} \big\}, \end{aligned} \tag{5.8}$$

for some constant K. In the last step we used dominated convergence theorem.

Now we can use the Lifshitz tails representation for the asymptotics of the a.s. non-random limiting IDS, $\nu(E)$, see (3.11), which implies:

$$\limsup_{n\to\infty} e^{an^{d/2}} \nu(1/n) \leq 1, \tag{5.9}$$

for $a > 0$. Since we assumed that $0 < \gamma < d/2$, there exists $n_0 < \infty$ such that by (5.8) and (5.9) for all $n > n_0$ we get:

$$\lim_{M\to\infty} \mathbb{P}\big(V_M^n\big) = 0.$$

This last result, along with (5.3) and (5.7), implies that:

$$\sum_{n>n_0} \mathbb{P}\left(\bigcup_{l\geq l_0} S_l^n \right) \leq \sum_{n>n_0} \sum_{k=1}^{\infty} \mathbb{P}\big(W_k^n\big). \tag{5.10}$$

Now, we show that the upper bound in (5.10) is finite. First we split the box $\Lambda_{[(ke^{\alpha n^\gamma})^{1/d}]_+}$ into $m(k,n)$ disjoints sub-cubes of the side $l(k,n)$, with the following choice of parameters:

$$m(k,n) := [kM_n]_+, \quad M_n := B^{-d/2} e^{\alpha n^\gamma} n^{-d/2},$$

$$l(k,n) := \frac{[(ke^{\alpha n^\gamma})^{1/d}]_+}{(m(k,n))^{1/d}}.$$

Here B is the constant that comes from Lemma 5.1. Now by the Dirichlet-Neumann inequality, see e.g. [22], Chap. XIII.15, we get:

$$h^D_{[(ke^{\alpha n^\gamma})^{1/d}]_+} \geq h^N_{[(ke^{\alpha n^\gamma})^{1/d}]_+} \geq \bigoplus_{j=1}^{m(k,n)} h^{j,N}_{l(k,n)}, \tag{5.11}$$

where $h^{j,N}_{l(k,n)}$ denotes the Schrödinger operator defined in the j-th sub-cube of the side $l(k,n)$, with Neumann boundary conditions. Note that, by the *positivity* of the random potential, we obtain:

$$E^{\omega,N}_{j,2} \geq \varepsilon^N_{j,2} \geq \frac{\pi}{l(k,n)^2} \geq \frac{1}{n}. \tag{5.12}$$

Here $E^{\omega,N}_{j,2}$ denotes the *second eigenvalue* of the operator $h^{j,N}_{l(k,n)}$, and $\varepsilon^N_{j,2}$ the *second eigenvalue* of $-\Delta^{j,N}_{l(k,n)}$, i.e. the kinetic-energy operator defined in the j-th sub-cube of the side $l(k,n)$ with the Neumann boundary conditions.

By (5.12), we know that to estimate the probability of the set (5.4) by using the Dirichlet-Neumann inequality (5.11), only the *ground state* of each operator $h^{j,N}_{l(k,n)}$ is relevant. Since the sub-cubes are *stochastically independent*, we have:

$$\mathbb{P}(W^n_k) \leq \mathbb{P}\{\omega : \sharp\{j : E^{\omega,N}_{j,1} < 1/n\} = k\} \leq {}^{m(k,n)}C_k q^k (1-q)^{m(k,n)-k} \leq {}^{m(k,n)}C_k q^k$$

with q being the probability $\mathbb{P}\{\omega : E^{\omega,N}_{j,1} < 1/n\}$. The latter can be estimated by Lemma 5.1. So, finally we obtain the upper bound:

$$\mathbb{P}(W^n_k) \leq {}^{m(k,n)}C_k \exp\{-kA(l(k,n))^d\}. \tag{5.13}$$

Using Stirling's inequalities, see [23], Chap. II.12:

$$(2\pi)^{1/2} n^{n+1/2} e^{-n} \leq n! \leq 2(2\pi)^{1/2} n^{n+1/2} e^{-n}$$

we can give an upper bound for the binomial coefficients ${}^{m(k,n)}C_k$ in the form:

$$\frac{2(2\pi)^{\frac{1}{2}}(kM_n+\delta)^{(kM_n+\delta+1/2)}\exp(-kM_n+\delta)}{(2\pi)k^{k+\frac{1}{2}}\exp(-k)\cdot(kM_n+\delta-k)^{(kM_n+\delta-k+1/2)}\exp(-kM_n+\delta-k)}, \tag{5.14}$$

where $\delta \geq 0$ is defined by:

$$m(k,n) = [kM_n]_+ = kM_n + \delta.$$

Then (5.14) implies the estimate:

$${}^{m(k,n)}C_k \leq K_1 \frac{(kM_n+\delta)^{kM_n+\delta+1/2}}{k^{k+\frac{1}{2}}(kM_n-k)^{kM_n+\delta-k+1/2}} \leq K_1 (M_n)^k \left(\frac{(1+\sigma_1)^{(kM_n+\delta+\frac{1}{2})}}{(1-\sigma_2)^{(kM_n+\delta+\frac{1}{2}-k)}}\right),$$

for some $K_1 > 0$ and

$$\sigma_1 := \delta(kM_n)^{-1}, \qquad \sigma_2 := M_n^{-1}.$$

Since $\delta/k < 1$ and $\sigma_{1,2} \to 0$ as $n \to \infty$, and also using the fact that $x \ln(1 + 1/x) \to 1$ as $x \to \infty$, we can find a constant $c > 0$ such that, for n large enough one gets the estimate:

$$^{m(k,n)}C_k \leq K_1(M_n)^k \left(\frac{(1 + M_n^{-1})^{(kM_n)}}{(1 - M_n^{-1})^{(kM_n - k)}} \right) \leq K_1(M_n)^k e^{ck}. \tag{5.15}$$

The side $l(k, n)$ of sub-cubes has a lower bound:

$$l(k,n) = \frac{[(ke^{\alpha n^\gamma})^{1/d}]_+}{(m(k,n))^{1/d}} \geq \frac{(ke^{\alpha n^\gamma})^{1/d}}{(ke^{\alpha n^\gamma}(Bn)^{-d/2} + \delta)^{1/d}} \geq \left(B^{d/2} n^{d/2} \frac{1}{1 + \sigma_1} \right)^{1/d}. \tag{5.16}$$

Combining (5.15), (5.16) and (5.13) we obtain a sufficient upper bound:

$$\begin{aligned}
\sum_{k \geq 1} \mathbb{P}\big(W_k^n\big) &\leq \sum_{k \geq 1} {}^{m(k,n)}C_k e^{-kAl^d(k,n)} \\
&\leq \sum_{k \geq 1} K_1(M_n)^k e^{ck} e^{-kAB^{d/2}n^{d/2}/(1+\sigma_1)} \\
&\leq K_2 \sum_{k \geq 1} \exp\big\{k\big(\alpha n^\gamma - (d/2)\ln(nB) + c - AB^{d/2}n^{d/2}\big)\big\} \\
&\leq K_3 \sum_{k \geq 1} \exp k\big(\alpha n^\gamma - AB^{d/2}n^{d/2} + K_4\big) \leq K_5 \exp\big(-K_6 n^{d/2}\big).
\end{aligned}$$

Here K_i are some finite, positive constants independent of k, n, l, for any n large enough. Now the lemma immediately follows from (5.10). □

Proof of Theorem 5.1 Let A_n to be the event:

$$A_n := \{\omega : \nu_l^\omega(1/n) > e^{-\alpha n^\gamma} \text{ for some } l\}. \tag{5.17}$$

By Lemma 5.2, we have:

$$\sum_{n \geq 1} \mathbb{P}\big(A_n\big) < \infty,$$

and therefore, by the Borel-Cantelli lemma one gets that with probability one, only a *finite* number of events A_n occur. In other words, there is a subset $\widetilde{\Omega} \subset \Omega$ of full measure, $\mathbb{P}(\widetilde{\Omega}) = 1$, such that for any $\omega \in \widetilde{\Omega}$ one can find a *finite* and independent on l number $n_0(\omega) < \infty$ for which, in contrast to (5.17), we have:

$$\nu_l^\omega(1/n) \leq e^{-\alpha n^\gamma}, \quad \text{for all } n > n_0(\omega) \text{ and for all } l \geq 1.$$

Define $\mathcal{E}(\omega) := 1/n_0(\omega)$. For any $E \leq \mathcal{E}(\omega)$, we can find $n \geq n_0(\omega)$ such that:

$$\frac{1}{2n} \leq E \leq \frac{1}{n},$$

and the theorem follows with the constant α modified by a factor $2^{-\gamma}$. □

6 On the Nature of the Generalized Condensates in the Luttinger-Sy Model

In this section, we study the van den Berg-Lewis-Pulé classification of generalized BE condensation (see discussion in Sect. 3) in a particular case of the so-called Luttinger-Sy model with point impurities [3]. Formally the single particle Hamiltonian for this model is

$$h_l^\omega = -\frac{1}{2}\Delta + a \sum_j \delta(x - x_j^\omega), \tag{6.1}$$

where the x_j's are distributed according to a Poisson law and $a = +\infty$.

We first recalls some definitions to make sense of this formal Hamiltonian. Let $u(x) \geq 0$, $x \in \mathbb{R}$, be a continuous function with a *compact* support called a (*repulsive*) single-impurity potential. Let $\{\mu_\lambda^\omega\}_{\omega\in\Omega}$ be the *random* Poisson measure on $\mathbb{R}$ with intensity $\lambda > 0$:

$$\mathbb{P}(\{\omega \in \Omega : \mu_\lambda^\omega(\Lambda) = n\}) = \frac{(\lambda|\Lambda|)^n}{n!} e^{-\lambda|\Lambda|}, \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \tag{6.2}$$

for any bounded Borel set $\Lambda \subset \mathbb{R}$. Then the non-negative random potential v^ω generated by the Poisson distributed local impurities has realizations

$$v^\omega(x) := \int_{\mathbb{R}} \mu_\lambda^\omega(dy) u(x - y) = \sum_{x_j^\omega \in X^\omega} u(x - x_j^\omega). \tag{6.3}$$

Here the random set X^ω corresponds to impurity positions $X^\omega = \{x_j^\omega\}_j \subset \mathbb{R}$, which are the atoms of the random point Poisson measure, i.e., $\sharp\{X^\omega \restriction \Lambda\} = \mu_\lambda^\omega(\Lambda)$ is the number of impurities in the set Λ. Since the expectation $\mathbb{E}(\nu_\lambda^\omega(\Lambda)) = \lambda|\Lambda|$, the parameter λ coincides with the *density* of impurities on $\mathbb{R}$.

Luttinger and Sy defined their model by restriction of the single-impurity potential to the case of point δ-potential with amplitude $a \to +\infty$. Then the corresponding random potential (6.3) takes the form:

$$v_a^\omega(x) := \int_{\mathbb{R}} \nu_\lambda^\omega(dy) a\delta(x - y) = a \sum_{x_j^\omega \in X^\omega} \delta(x - x_j^\omega). \tag{6.4}$$

Now the self-adjoint one-particle random Schrödinger operator $h_a^\omega := h^0 \dotplus v_a^\omega$ is defined in the sense of the sum of quadratic forms (2.2). The strong resolvent limit $h_{LS}^\omega := s.r. \lim_{a\to+\infty} h_a^\omega$ is the Luttinger-Sy model.

Since X^ω generates a set of intervals $\{I_j^\omega := (x_{j-1}^\omega, x_j^\omega)\}_j$ of lengths $\{L_j^\omega := x_j^\omega - x_{j-1}^\omega\}_j$, one gets decompositions of the one-particle Luttinger-Sy Hamiltonian:

$$h_{LS}^\omega = \bigoplus_j h_D(I_j^\omega), \quad \text{dom}(h_{LS}^\omega) \subset \bigoplus_j L^2(I_j^\omega), \quad \omega \in \Omega, \tag{6.5}$$

into random disjoint free Schrödinger operators $\{h_D(I_j^\omega)\}_{j,\omega}$ with *Dirichlet* boundary conditions at the end-points of intervals $\{I_j^\omega\}_j$. Then the Dirichlet restriction $h_{l,D}^\omega$ of the Hamiltonian h_{LS}^ω to a fixed interval $\Lambda_l = (-l/2, l/2)$ and the corresponding change of notations are evident: e.g., $\{I_j^\omega\}_j \mapsto \{I_j^\omega\}_{j=1}^{M^l(\omega)}$, where $M^l(\omega)$ is total number of subintervals in Λ_l corresponding to the set X^ω. For rigorous definitions and some results concerning this model we refer the reader to [5].

Since this particular choice of random potential is able to produce Lifhsitz tails in the sense of (3.11), see Proposition 3.2 in [5], it follows that such a model exhibits a generalized BEC in random eigenstates, see (3.4). In fact, it was shown in [5] that *only* the random ground state $\phi_1^{\omega,l}$ of $h_{l,D}^{\omega}$ is *macroscopically* occupied. In our notations this means that

$$\lim_{l\to\infty}\frac{1}{l}\langle N_l(\phi_1^{\omega,l})\rangle_l = \begin{cases}0 & \text{if } \overline{\rho}<\rho_c,\\ \overline{\rho}-\rho_c & \text{if } \overline{\rho}\geq\rho_c,\end{cases}$$
$$\lim_{l\to\infty}\frac{1}{l}\langle N_l(\phi_i^{\omega,l})\rangle_l = 0, \quad \text{for all } i>1. \tag{6.6}$$

According to the van den Berg-Lewis-Pulé classification this corresponds to the *type I* Bose-condensation in the random eigenstates $\{\phi_i^{\omega}\}_{i\geq 1}$.

Following the line of reasoning of Sect. 4, we now consider the corresponding BEC in the kinetic-energy eigenstates. We retain the notation used in that section and explain briefly the minor changes required in the application of our method to the Luttinger-Sy model.

We first state the equivalent of Theorem 4.1 for this particular model.

Theorem 6.1 *Theorem 4.1 holds with the function g defined as follows*

$$g(k)=\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}\mathrm{d}x\, e^{ikx}\sum_{n\geq 1}e^{n\beta\mu_\infty}\frac{e^{-\|x\|^2(1/2n\beta)}}{(2\pi n\beta)^{d/2}}$$
$$\times\int_{\Omega^{n\beta}_{(0,x)}} w^{n\beta}(\mathrm{d}\xi)\exp\Big(-\lambda\Big(\sup_s\xi(s)-\inf_s\xi(s)\Big)\Big).$$

The scheme of the proof is the same as above, cf. Sects. 4 and 5. First, we note that Lemmas 4.1 and 4.2 apply immediately. The positivity of the random potential has to be understood in terms of quadratic forms, see (2.4).

Before continuing, we need to highlight a minor change concerning the *finite-volume* Lifshitz tails arguments. Although the Theorem 5.1 is valid for the Luttinger-Sy model, its proof (see Sect. 5) requires a minor modification, as the assumption of Lemma 5.1 is clearly not satisfied for the case of singular potentials. However, by direct calculation we can obtain the same estimate with the constant $B=\pi^2/4$ in (5.1). First, suppose that there is at least one impurity in the box, then the eigenvalues will be of the form (for some j)

$$(n^2\pi^2)/(L_j^{\omega})^2, \quad n=1,2,\dots$$

if I_j^{ω} is an inner interval (that is, its two endpoints correspond to impurities), and

$$((n+1/2)^2\pi^2)/(L_j^{\omega})^2, \quad n=0,1,2,\dots.$$

If I_j^{ω} is an outer interval (that is, one endpoint corresponds to an impurity, and the other one to the boundary of Λ_l). Therefore, $E_1^{\omega,l,N}\geq B/l^2$ since obviously $L_j^{\omega}<l$. Now, if there is no impurity in the box Λ_l, then $E_1^{\omega,l,N}=0<B/l^2$. But due to the Poisson distribution (6.2) this happens with probability $e^{-\lambda l}$, proving the same estimate as in Lemma 5.1.

With this last observation, the proof of the Theorem 5.1 in Sect. 5 can be carried out verbatim, without any further changes.

Our next step is to split the measure $\tilde{m}_l$ into two, $\tilde{m}_l^{(1)}$ and $\tilde{m}_l^{(2)}$, see (4.9), and prove the statement equivalent to the Theorem 4.2.

Theorem 6.2 *For any $d \geq 1$, the sequence of Laplace transforms of the measures $\tilde{m}_l^{(1)}$:*

$$f_l(t;\beta,\mu_l) := \int_{\mathbb{R}} \tilde{m}_l^{(1)}(\mathrm{d}\varepsilon)\, e^{-t\varepsilon}$$

converges for any $t > 0$ to a (non-random) limit $f(t;\beta,\mu_\infty)$, which is given by:

$$f(t;\beta,\mu_\infty) = \sum_{n\geq 1} e^{n\beta\mu_\infty} \int_{\mathbb{R}^d} \mathrm{d}x \frac{e^{-\|x\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \times \int_{\Omega^{n\beta}_{(0,x)}} w^{n\beta}(\mathrm{d}\xi) \exp\Big(-\lambda\Big(\sup_s \xi(s) - \inf_s \xi(s)\Big)\Big).$$

Proof We follow the proof of Theorem 4.2, using the same notation. The uniform convergence is obtained the same way, since the bounds (4.14), (4.15), and (4.16) are also valid in this case. As in (4.20), we can use the ergodic theorem to obtain:

$$\lim_{l\to\infty} a_l(n) = e^{n\beta\mu_\infty} \mathbb{E}_\omega \int_{\mathbb{R}} \mathrm{d}x \frac{e^{-\|x\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \sum_j \int_{\Omega^{n\beta}_{(0,x)}} w^{n\beta}(\mathrm{d}\xi) \chi_{I_j{}^\omega, n\beta}(\xi). \qquad (6.7)$$

We have used the fact that the Dirichlet boundary conditions at the impurities split up the space $\mathcal{H}_l$ into a direct sum of Hilbert spaces (see (6.5)). This can be seen from the expression

$$\lim_{l\to\infty} a_l(n) = e^{n\beta\mu_\infty} \int_{\mathbb{R}} \mathrm{d}x \frac{e^{-\|x\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \mathbb{E}_\omega \int_{\Omega^{n\beta}_{(0,x)}} w^{n\beta}(\mathrm{d}\xi) e^{-\int_0^{n\beta} \mathrm{d}s a \sum_{x_j^\omega \in X^\omega} \delta(\xi(s) - x_j^\omega)}$$

by formally putting the amplitude, a, of the point impurities (6.4) equal to $+\infty$. Because of the characteristic functions $\chi_{I_j^\omega, n\beta}$, which constrain the paths ξ to remain in the interval I_j^ω in time $n\beta$, the sum in (6.7) reduces to only *one* term:

$$\lim_{l\to\infty} a_l(n) = e^{n\beta\mu_\infty} \int_{\mathbb{R}} \mathrm{d}x \frac{e^{-\|x\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \mathbb{E}_\omega \int_{\Omega^{n\beta}_{(0,x)}} w^{n\beta}(\mathrm{d}\xi) \chi_{(a_\omega, b_\omega), n\beta}(\xi), \qquad (6.8)$$

where (a_ω, b_ω), is the interval among the I_j^ω's which contains 0.

The expression in (6.8) can be simplified further by computing the expectation $\mathbb{E}_\omega$ explicitly.

First, note that the Poisson impurity positions: a_ω, b_ω are independent random variables and by definition, a_ω is negative while b_ω is positive. For the random variable b_ω the distribution function is:

$$\mathbb{P}\,(b_\omega < b) := \mathbb{P}\{(0,b) \text{ contains at least one impurity}\} = 1 - e^{-\lambda b},$$

and therefore its probability density is $\lambda e^{-\lambda b}$ on $(0,\infty)$. Similarly for a_ω one gets:

$$\mathbb{P}\,(a_\omega < a) := \mathbb{P}\{(a,0) \text{ contains no impurities}\} = e^{-\lambda|a|} = e^{\lambda a},$$

and thus its density is $\lambda e^{\lambda a}$ on $(-\infty, 0)$. Using these distributions in (6.8) we obtain:

$$\lim_{l\to\infty} a_l(n) = e^{n\beta\mu_\infty} \lambda^2 \int_{-\infty}^0 \mathrm{d}a e^{\lambda a} \int_0^\infty \mathrm{d}b e^{-\lambda b} \int_{\mathbb{R}} \mathrm{d}x \frac{e^{-\|x\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}}$$

$$\times \int_{\Omega^{n\beta}_{(0,x)}} w^{n\beta}(\mathrm{d}\xi)\chi_{(a,b)}(\xi)$$

$$= e^{n\beta\mu_\infty}\lambda^2 \int_{-\infty}^{0} \mathrm{d}a e^{\lambda a} \int_0^\infty \mathrm{d}b e^{-\lambda b} \int_{\mathbb{R}} \mathrm{d}x \frac{e^{-\|x\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}}$$

$$\times \int_{\Omega^{n\beta}_{(0,x)}} w^{n\beta}(\mathrm{d}\xi)\mathbf{1}\Big(\sup_s(\xi(s)) \le b\Big)\mathbf{1}\Big(\inf_s(\xi(s)) \ge a\Big)$$

$$= e^{n\beta\mu_\infty}\lambda^2 \int_{\mathbb{R}} \mathrm{d}x \frac{e^{-\|x\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}}$$

$$\times \int_{\Omega^{n\beta}_{(0,x)}} w^{n\beta}(\mathrm{d}\xi) \int_{-\infty}^{\inf_s(\xi(s))} \mathrm{d}a e^{\lambda a} \int_{\sup_s(\xi(s))}^{\infty} \mathrm{d}b e^{-\lambda b},$$

and the Theorem 6.2 follows by explicit computation of the last two integrals. □

Proof of Theorem 6.1 Having proved Theorem 6.2, it is now straightforward to derive the analogue of Corollary 4.1 for the Luttinger-Sy model. Note also that the Corollary 4.2 remains unchanged, since only the uniform convergence was used. With these results, the proof of Theorem 6.1 follows in the same way as for Theorem 4.1. □

We have proved, in Theorem 6.1, that the Luttinger-Sy model exhibits g-BEC in the kinetic energy states. But, in this particular case, we can go further and determine the particular *type* of g-BEC in the kinetic energy states. Recall that the g-BEC in the *random* eigenstates is only in the *ground* state, that is, of the *type* I, see (6.6) and [5] for a comprehensive review. Here we shall show that the g-BEC in the kinetic-energy eigenstates is in fact of the *type* III, namely:

Theorem 6.3 *In the Luttinger-Sy model none of the kinetic-energy eigenstates is macroscopically occupied*:

$$\lim_{l\to\infty} \frac{1}{l}\langle N_l(\psi_k)\rangle_l = 0 \quad \text{for all } k \in \Lambda_l^*,$$

even though for $\overline{\rho} > \rho_c$ there is a generalized BEC.

To prove this theorem we shall exploit the finite-volume localization properties of the random eigenfunctions $\phi_i^{\omega,l}$ of the Hamiltonian $h^\omega_{l,D}$. Since the impurities split up the box Λ_l into a finite number $M^l(\omega)$ of sub-intervals $\{I_j^\omega\}_{j=1}^{M^l(\omega)}$, by virtue of the corresponding orthogonal decomposition of $h^\omega_{l,D}$, cf. (6.5), the normalized random eigenfunctions $\phi_s^{\omega,l}$ are in fact *sine-waves* with supports in each of these sub-intervals and thus satisfy:

$$|\phi_s^{\omega,l}(x)| < \sqrt{\frac{2}{L^\omega_{j_s}}}\mathbf{1}_{I^\omega_{j_s}}(x), \quad 1 \le j_s \le M^l(\omega). \tag{6.9}$$

We require an estimate of the size L_j^ω of these random sub-intervals, which we obtain in the following lemma.

Lemma 6.1 *Let* $\lambda > 0$ *be a mean concentration of the point Poisson impurities on* $\mathbb{R}$. *Then eigenfunctions* ϕ_j^ω *are localized in sub-intervals of logarithmic size, in the sense that for any* $\kappa > 4$, *one has a.s. the estimate*:

$$\limsup_{l\to\infty} \frac{\max_{1\le j\le M^l(\omega)} L_j^\omega}{\ln l} \le \frac{\kappa}{\lambda}.$$

Proof Define the set

$$S_l := \left\{\omega : \max_{1\le j\le M^l(\omega)} L_j^\omega > \frac{\kappa}{\lambda}\ln l\right\}.$$

Let $n := [2\lambda l/(\kappa \ln l)]_+$, and define a new box:

$$\widetilde{\Lambda}_l := \left[-\frac{n}{2}\left(\frac{\kappa}{2\lambda}\ln l\right), \frac{n}{2}\left(\frac{\kappa}{2\lambda}\ln l\right)\right] \supset \Lambda_l.$$

Split this bigger box into n identical disjoints intervals $\{I_m^l\}_{m=1}^n$ of size $\kappa(2\lambda)^{-1}\ln l$. If $\omega \in S_l$, then there exists at least one empty interval I_m^l (interval without any impurities), and therefore the set

$$S_l \subset \bigcup_{1\le m\le n} \{\omega : I_m^l \text{ is empty}\}.$$

By the Poisson distribution (6.2), the probability for the interval I_m^l to be empty depends only on its size, and thus

$$\mathbb{P}(S_l) \le n \exp\left(-\lambda\frac{\kappa}{2\lambda}\ln l\right) \le \left[\frac{2\lambda l}{\kappa \ln l}\right]_+ l^{-\kappa/2}.$$

Since we choose $\kappa > 4$, it follows that

$$\sum_{l\ge 1}\mathbb{P}(S_l) < \infty.$$

Therefore, by the Borel-Cantelli lemma, there exists a subset $\widetilde{\Omega} \subset \Omega$ of full measure, $\mathbb{P}(\widetilde{\Omega}) = 1$, such that for each $\omega \in \widetilde{\Omega}$ one can find $l_0(\omega) < \infty$ with

$$\mathbb{P}\left\{\omega : \max_{1\le j\le M^l(\omega)} L_j^\omega \le \frac{\kappa}{\lambda}\ln l\right\} = 1,$$

for all $l \ge l_0(\omega)$. □

Now we can prove the main statement of this section.

Proof of Theorem 6.3 The atom of the measure $\tilde{m}$ has already been established in Theorem 6.1. Concerning the macroscopic occupation of a single state, we have

$$\frac{1}{l}\langle N_l(\psi_k)\rangle_l = \frac{1}{l}\sum_i |(\phi_i^{\omega,l}, \psi_k)|^2 \langle N_l(\phi_i^{\omega,l})\rangle_l$$

$$= \frac{1}{l}\sum_i \langle N_l(\phi_i^{\omega,l})\rangle_l \left|\int_{\Lambda_l} \mathrm{d}x \overline{\psi}_k(x)\phi_i^{\omega,l}(x)\right|^2$$

$$\leq \frac{1}{l}\sum_i \langle N_l(\phi_i^{\omega,l})\rangle_l \frac{1}{l}\left(\int_{\Lambda_l} \mathrm{d}x |\phi_i^{\omega,l}(x)|\right)^2,$$

where in the last step we have used the bound $|\psi_k| \leq 1/\sqrt{l}$. Therefore, by (6.9) and Lemma 6.1, we obtain a.s. the following estimate:

$$\frac{1}{l}\langle N_l(\psi_k)\rangle_l \leq \frac{1}{l}\sum_i \langle N_l(\phi_i^{\omega,l})\rangle_l \frac{1}{l}\frac{\kappa}{\lambda}\ln l,$$

which is valid for large enough l and for any $\kappa > 4$. The theorem then follows by taking the thermodynamic limit. □

7 Application to Weak (Scaled) Non-random Potentials

It is known for a long time, see e.g. [24, 25], that BEC can be enhanced in low-dimensional systems by imposing a weak (scaled) external potential. Recently this was a subject of a new approach based on the Random Boson Point Field method [26]. In this section, we show that, with some minor modifications our method can be extended to cover also the case of these scaled *non-random* potentials.

Let v be a non-negative, continuous real-valued function defined on the closed unit cube $\overline{\Lambda_1} \subset \mathbb{R}^d$. The *one-particle* Schrödinger operator with a *weak* (*scaled*) external potential in a box Λ_l is defined by:

$$h_l = -\frac{1}{2}\Delta_D + v(x_1/l, \ldots, x_d/l). \tag{7.1}$$

Let $\{\varphi_i^l, E_i^l\}_{i\geq 1}$ be the set of orthonormal eigenvectors and corresponding eigenvalues of the operator (7.1). As usual we put $E_1 \leq E_2 \leq \cdots$ by convention. The many-body Hamiltonian for the perfect Bose gas is defined in the same way as in Sect. 2. We keep the notations m and $\tilde{m}$ for the occupation measures of the eigenstates $\{\varphi_i^l\}_{i\geq 1}$ and of the kinetic-energy states respectively. We denote the *integrated density of states* (IDS) of the Schrödinger operator (7.1) by ν_l, and by $\nu = \lim_{l\to\infty} \nu_l$ its weak limit. We assume that the first eigenvalue $E_1^l \to 0$ as $l \to \infty$, which is the case, when e.g. $v(0) = 0$. This assumption is equivalent to condition (iii), Sect. 2. It ensures that for a given mean particle density $\overline{\rho}$ the chemical potential $\mu_\infty(\beta, \overline{\rho})$ satisfies the relation (3.6), where $\overline{\mu} := \overline{\mu}(\beta, \overline{\rho})$ is a (unique) solution of the equation [24]:

$$\overline{\rho} = \sum_{n\geq 1} \frac{1}{(2\pi n\beta)^{d/2}} \int_{\Lambda_1} \mathrm{d}x e^{n\beta(\mu - v(x))} = \int_{[0,\infty)} \nu^0(\mathrm{d}E) \int_{\Lambda_1} \mathrm{d}x \left(e^{\beta(E+v(x)-\mu)} - 1\right)^{-1}, \tag{7.2}$$

for $\overline{\rho} \leq \rho_c$, where the boson critical density is given by:

$$\rho_c = \sum_{n\geq 1} \frac{1}{(2\pi n\beta)^{d/2}} \int_{\Lambda_1} \mathrm{d}x e^{-n\beta v(x)} = \int_{[0,\infty)} \nu^0(\mathrm{d}E) \int_{\Lambda_1} \mathrm{d}x \left(e^{\beta(E+v(x))} - 1\right)^{-1}. \tag{7.3}$$

Here ν^0 is the IDS (3.9) of the kinetic-energy operator (2.1). In particular the value $\rho_c = \infty$ is allowed in (7.3). If $\rho_c < \infty$, the existence of a generalized BEC in the states $\{\varphi_i^l\}_{i\geq 1}$ follows by the same arguments as in Sect. 3. For example, the choice: $v(x) = |x|$, makes the critical density finite even in dimension one, see e.g. [24].

Now, we prove the statements equivalent to the Theorem 4.1:

Theorem 7.1 *The sequence $\{\tilde{m}_l\}_{l\geq 1}$ of the one-particle kinetic states occupation measures has a weak limit $\tilde{m}$ given by*:

$$\tilde{m}(\mathrm{d}\varepsilon) = \begin{cases} (\overline{\rho} - \rho_c)\delta_0(\mathrm{d}\varepsilon) + F(\varepsilon)\nu^0(\mathrm{d}\varepsilon), & \text{if } \overline{\rho} \geq \rho_c, \\ F(\varepsilon)\nu^0(\mathrm{d}\varepsilon), & \text{if } \overline{\rho} < \rho_c, \end{cases}$$

where the density $F(\varepsilon)$ is defined by:

$$F(\varepsilon) = \int_{\Lambda_1} \mathrm{d}x \left(e^{\beta(\varepsilon + v(x) - \mu_\infty)} - 1\right)^{-1},$$

and $\mu_\infty := \mu_\infty(\beta, \overline{\rho})$ satisfies the relation (3.6).

We note the similarity of this result with the free Bose gas. Indeed, the kinetic-energy states occupation measure density is reduced to the free gas one, with the energy shifted by the external potential v and then *averaged* over the unit cube.

The proof requires the same tools as in the random case. As before, we split the occupation measure into two parts:

$$\begin{aligned} \tilde{m}_l &= \tilde{m}_l^{(1)} + \tilde{m}_l^{(2)} \quad \text{with} \\ \tilde{m}_l^{(1)}(A) &:= \sum_{n\geq 1} \frac{1}{V_l} \operatorname{Tr} P_A(e^{-n\beta(h_l - \mu_l)})\mathbf{1}(\mu_l \leq 1/n), \\ \tilde{m}_l^{(2)}(A) &:= \sum_{n\geq 1} \frac{1}{V_l} \operatorname{Tr} P_A(e^{-n\beta(h_l - \mu_l)})\mathbf{1}(\mu_l > 1/n), \end{aligned}$$

and we prove the following statement:

Theorem 7.2 *The sequence of measures $\tilde{m}_l^{(1)}$ converges weakly to a measure $\tilde{m}^{(1)}$, which is absolutely continuous with respect to ν^0 with density $F(\varepsilon)$ given by*:

$$F(\varepsilon) = \int_{\Lambda_1} \mathrm{d}x \left(e^{\beta(\varepsilon + v(x) - \mu_\infty)} - 1\right)^{-1}.$$

Proof We follow the line of reasoning of the proof of Theorem 4.2. Let $g_l(t; \beta, \mu_l)$ be the Laplace transform of the measure $\tilde{m}_l^{(1)}$:

$$\begin{aligned} g_l(t; \beta, \mu_l) &= \int_{\mathbb{R}} m_l^{(1)}(\mathrm{d}\varepsilon) e^{-t\varepsilon} \\ &= \sum_{n\geq 1} \frac{1}{V_l} \operatorname{Tr} e^{-th_l^0} \left(e^{-n\beta(h_l - \mu_l)}\right)\mathbf{1}(\mu_l \leq 1/n). \end{aligned} \tag{7.4}$$

Again, our aim is to show the uniform convergence of the sum over n with respect to l. Let

$$\begin{aligned} a_l(n) &:= \frac{1}{V_l}\operatorname{Tr} e^{-th_l^0} e^{-n\beta(h_l-\mu_l)}\mathbf{1}(\mu_l \le 1/n) \\ &\le \frac{1}{V_l}\operatorname{Tr} e^{-n\beta(h_l-\mu_l)}\mathbf{1}(\mu_l \le 1/n). \end{aligned} \tag{7.5}$$

Then for $\overline{\rho} < \rho_c$ we can apply a similar argument as for the random case, since the estimate $\mu_l < \mu_\infty/2 < 0$ still holds, to obtain:

$$a_l(n) \le e^{n\beta\mu_\infty/2}\int_{[0,\infty)} e^{-\beta\varepsilon}\nu_l(\mathrm{d}\varepsilon) \le K_1 e^{n\beta\mu_\infty/2}.$$

If $\overline{\rho} \ge \rho_c$, then $\mu_l \le 1/n$ in (7.5) implies that:

$$a_l(n) \le e^{\beta}\sum_i e^{-n\beta E_i^l} \le \frac{e^{\beta}}{(2\pi n\beta)^{d/2}}\int_{\Lambda_1}\mathrm{d}x\, e^{-n\beta v(x)},$$

where the last estimate can be found in [24] or [25]. Now the uniform convergence for the sequence $a_l(n)$ follows from (7.3), since we assumed that $\rho_c < \infty$. The latter implies also that for $\overline{\rho} \ge \rho_c$, $\mu_\infty(\beta,\overline{\rho}) = 0$. Thus, we can take the limit of the Laplace transform (7.4) term by term, that is:

$$\begin{aligned} \lim_{l\to\infty} a_l(n) &= \lim_{l\to\infty}\frac{1}{V_l}\operatorname{Tr} e^{-th_l^0} e^{-n\beta(h_l-\mu_l)}\mathbf{1}(\mu_l \le 1/n) \\ &= \lim_{l\to\infty}\frac{1}{V_l}\int_{\Lambda_l}\int_{\Lambda_l}\mathrm{d}x\,\mathrm{d}x'\, e^{-th_l^0}(x,x')e^{-n\beta(h_l-\mu_l)}(x',x) \\ &= e^{n\beta\mu_\infty}\lim_{l\to\infty}\frac{1}{V_l}\int_{\Lambda_l}\int_{\Lambda_l}\mathrm{d}x\,\mathrm{d}x'\frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \\ &\quad\times\int_{\Omega^t_{(x,x')}} w^t(\mathrm{d}\xi')\chi_{\Lambda_l,t}(\xi')\int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi)e^{-\int_0^{n\beta}\mathrm{d}s\, v(\xi(s)/l)}\chi_{\Lambda_l,n\beta}(\xi). \end{aligned} \tag{7.6}$$

Here we have used the Feynman-Kac representation for free $e^{-th_l^0}(x,y)$ and for non-free $e^{-\beta h_l}(x,y)$ Gibbs semi-group kernels, where w^T stands for the *normalized* Wiener measure on the path-space $\Omega^T_{(x,y)}$, see Sect. 4.1.

Note that by Lemma A.2, which demands only the *non-negativity* of the potential v, we obtain for (7.6) the representation:

$$\begin{aligned} &\lim_{l\to\infty}\frac{1}{V_l}\operatorname{Tr} e^{-th_l^0} e^{-n\beta(h_l-\mu_l)} \\ &\quad= e^{n\beta\mu_\infty}\lim_{l\to\infty}\int_{\mathbb{R}^d}\mathrm{d}x\frac{1}{V_l}\int_{\Lambda_l}\mathrm{d}x'\frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \\ &\quad\quad\times\int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi)e^{-\int_0^{n\beta}\mathrm{d}s\, v(\xi(s)/l)}. \end{aligned} \tag{7.7}$$

Now we express the trajectories ξ in terms of *Brownian bridges* $\alpha(\tau) \in \tilde{\Omega}$, $0 \le \tau \le 1$, we denote the corresponding measure by D. Letting $\tilde{x} = x'/l$, we obtain:

$$\begin{aligned}
&\lim_{l\to\infty} \frac{1}{V_l} \operatorname{Tr} e^{-th_l^0} e^{-n\beta(h_l-\mu_l)} \\
&= e^{n\beta\mu_\infty} \lim_{l\to\infty} \int_{\mathbb{R}^d} \mathrm{d}x \int_{\Lambda_1} \mathrm{d}\tilde{x} \frac{e^{-\|x-l\tilde{x}\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \\
&\quad \times \int_{\tilde{\Omega}} D(\mathrm{d}\alpha) \exp\left(-\int_0^{n\beta} \mathrm{d}s\, v\left[\left(1-\frac{s}{n\beta}\right)\tilde{x} + \frac{s}{n\beta}(x/l) + \frac{\sqrt{n\beta}}{l}\alpha(s/n\beta)\right]\right).
\end{aligned}$$

Since the integration with respect to x is now over the whole space, we let $y = x - l\tilde{x}$ to get

$$\begin{aligned}
&\lim_{l\to\infty} \frac{1}{V_l} \operatorname{Tr} e^{-th_l^0} e^{-n\beta(h_l-\mu_l)} \\
&= e^{n\beta\mu_\infty} \lim_{l\to\infty} \int_{\mathbb{R}^d} \mathrm{d}y \int_{\Lambda_1} \mathrm{d}\tilde{x} \frac{e^{-\|y\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \\
&\quad \times \int_{\tilde{\Omega}} D(\mathrm{d}\alpha) \exp\left(-\int_0^{n\beta} \mathrm{d}s\, v\left(\tilde{x} + \frac{s}{n\beta}(y/l) + \frac{\sqrt{n\beta}}{l}\alpha(s/n\beta)\right)\right) \\
&= e^{n\beta\mu_\infty} \int_{\mathbb{R}^d} \mathrm{d}y \frac{e^{-\|y\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \int_{\Lambda_1} \mathrm{d}\tilde{x} e^{-n\beta v(\tilde{x})},
\end{aligned}$$

where the last step follows from dominated convergence. Therefore, we obtain by (7.4) the following expression for the limiting Laplace transform:

$$\lim_{l\to\infty} g_l(t;\beta,\mu_l) = \sum_{n\ge 1} e^{-n\beta(E-\mu_\infty)} \frac{1}{(2\pi(n\beta+t))^{d/2}} \int_{\Lambda_1} \mathrm{d}x e^{-n\beta v(x)}.$$

It is now straightforward to invert this Laplace transform (for each term of the sum), to find that:

$$F(E)\nu^0(\mathrm{d}E) = \lim_{l\to\infty} \tilde{m}_l^1(\mathrm{d}E) = \sum_{n\ge 1} e^{-n\beta(E-\mu_\infty)} \left(\int_{\Lambda_1} \mathrm{d}x e^{-n\beta v(x)}\right) \nu^0(\mathrm{d}E).$$

The theorem then follows by Fubini's theorem. □

Proof of Theorem 7.1 The proof of Theorem 4.1 can be applied directly. Note that Lemmas 4.1, 4.2 are still valid, since (as we emphasized in Remarks 4.1, 4.2), their proofs require only the non-negativity of the external potential. Similarly, Corollary 4.2 now can be used directly, since we have proved Theorem 7.2. □

Appendix A: Brownian Paths

In this section, we first give an upper estimate of the probability of a Brownian path to leave some spatial domain, cf. e.g. [27] and the references quoted therein.

Lemma A.1 *Let the set*

$$\Omega^T_{(x,x')} := \{\xi(\tau) : \xi(0) = x, \xi(T) = x'\}$$

be continuous trajectories from x to x' with the proper time $0 \le \tau \le T$, and with the normalized Wiener measure w^T on it. Let x, x' be in Λ_l, and $\chi_{\Lambda_l,T}(\xi)$ the characteristic function over $\Omega^T_{(x,x')}$ of trajectories ξ staying in Λ_l for all $0 \le \tau \le T$. Then one gets the estimate:

$$\int_{\Omega^T_{(x,x')}} w^T(\mathrm{d}\xi)\big(1 - \chi_{\Lambda_l,T}(\xi)\big) \le e^{-C(T)(\min\{d(x,\partial\Lambda_l),d(x',\partial\Lambda_l)\})^2}. \tag{A.1}$$

Proof Define a *Brownian bridge* $\alpha(s), 0 \le s \le 1$ by:

$$\xi(t) = (1 - \tau/T)x + \tau/T\ x' + \sqrt{T}\alpha(\tau/T).$$

Let us consider first the one dimensional case, i.e. $\Lambda_l = [-l/2, l/2]$. Without loss of generality, we can assume that:

$$d(x, \partial\Lambda_l) \le d(x', \partial\Lambda_l).$$

Suppose that $x > 0$, then we have:

$$-x \le x' \le x \quad \text{and} \quad d(x, \partial\Lambda_l) = l/2 - x.$$

Assume that the path ξ leaves the box on the right-hand side. Then, for some t, we have:

$$\begin{aligned} \xi(t) &> \frac{l}{2}, \\ \alpha(t/T) &> \frac{1}{\sqrt{T}}\left(\frac{l}{2} + (t/T - 1)x - \frac{t}{T}x'\right), \\ \alpha(t/T) &> \frac{1}{\sqrt{T}}\left(\frac{l}{2} + (t/T - 1)x - \frac{t}{T}x\right) = \frac{1}{\sqrt{T}}d(x, \partial\Lambda_l). \end{aligned} \tag{A.2}$$

The case, when ξ leaves the box on the left-hand side can be treated similarly.

Let $x < 0$, then we have:

$$x \le x' \le -x \quad \text{and} \quad d(x, \partial\Lambda_l) = l/2 + x.$$

Again, assume that the path leaves the box on the right hand-side. Then, for some t, we have:

$$\begin{aligned} \xi(t) &> \frac{l}{2}, \\ \alpha(t/T) &> \frac{1}{\sqrt{T}}\left(\frac{l}{2} + (t/T - 1)x - \frac{t}{T}x'\right), \\ \alpha(t/T) &> \frac{1}{\sqrt{T}}\left(\frac{l}{2} - (t/T - 1)x' - \frac{t}{T}x'\right) \ge \frac{1}{\sqrt{T}}d(x, \partial\Lambda_l). \end{aligned} \tag{A.3}$$

The case, when ξ leaves the box on the left hand-side can be considered similarly. The relations (A.2), (A.3) imply that if ξ leaves the box Λ_l in one dimension, then the Brownian bridge α must satisfy the inequality:

$$\sup_t |\alpha(t/T)| > C(T)\min\{d(x,\partial\Lambda_l), d(x',\partial\Lambda_l)\}, \tag{A.4}$$

for some constant $C(T)$.

This observation can easily be extended to higher dimensions, when $x := (x_1,\dots,x_d)$ and $\alpha(s) := (\alpha_1(s),\dots,\alpha_d(s))$. Now, if ξ leaves the (d-dimensional) box Λ_l, there exists at least one i such that similar to (A.4):

$$\sup_t |\alpha_i(t/T)| > C(T)\min\{d(x_i,\partial_i\Lambda_l), d(x_i',\partial_i\Lambda_l)\},$$

where we denote $d(x_i,\partial_i\Lambda_l) := \min\{l/2 - x_i, l/2 + x_i\}$. Now, since Λ_l are cubes, we get $d(x_i,\partial_i\Lambda_l) \ge d(x,\partial\Lambda_l)$ for any $x \in \Lambda_l$. Then we obtain:

$$\begin{aligned} \|\alpha(t/T)\| &> |\alpha_i(t/T)|, \quad i = 1,\dots,d, \\ \sup_t \|\alpha(t/T)\| &> \max_i \sup_t |\alpha_i(t/T)|, \\ \sup_t \|\alpha(t/T)\| &> C(T)\min\{d(x_i,\partial_i\Lambda_l), d(x_i',\partial_i\Lambda_l)\} \\ &\ge C(T)\min\{d(x,\partial\Lambda_l), d(x',\partial\Lambda_l)\}. \end{aligned} \tag{A.5}$$

Therefore, the probability for the path ξ to leave the box is dominated by the probability for the one-dimensional Brownian bridge α to satisfy (A.5). The latter we can estimate using the following result from [27]:

$$\mathbb{P}\Big(\sup_s \alpha(s) > x\Big) \ge A e^{-Cx^2}$$

valid for some positive constants A, C, which implies the bound (A.1). □

Now we establish a result, that we use in the proof of Theorem 4.2:

Lemma A.2 *Let* $K^t_{\omega,l}(x,x')$, $K^t_{0,l}(x,x')$, $K^t_0(x,x')$ *be the kernels of operators* $\exp(-th^\omega_l)$, $\exp(-th^0_l)$, *and* $\exp(-t\Delta/2)$ *respectively. Then*

$$\begin{aligned} &\lim_{l\to\infty}\frac{1}{V_l}\int_{\Lambda_l}\int_{\Lambda_l} \mathrm{d}x\mathrm{d}x' K^t_{0,l}(x,x')K^{n\beta}_{\omega,l}(x',x) \\ &\quad = \lim_{l\to\infty}\int_{\mathbb{R}^d}\mathrm{d}x\frac{1}{V_l}\int_{\Lambda_l}\mathrm{d}x' K_0^{t+n\beta}(x,x')\int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi)e^{-\int_0^{n\beta}\mathrm{d}s\,v^\omega(\xi(s))}. \end{aligned} \tag{A.6}$$

Proof By the Feynman-Kac representation, we obtain:

$$\begin{aligned} &\lim_{l\to\infty}\frac{1}{V_l}\int_{\Lambda_l}\int_{\Lambda_l} \mathrm{d}x\,\mathrm{d}x' K^t_{0,l}(x,x')K^{n\beta}_{\omega,l}(x',x) \\ &\quad = \lim_{l\to\infty}\frac{1}{V_l}\int_{\Lambda_l}\int_{\Lambda_l}\mathrm{d}x\,\mathrm{d}x'\frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}} \\ &\qquad \times \int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi)e^{-\int_0^{n\beta}\mathrm{d}s\,v^\omega(\xi(s))}\chi_{\Lambda_l,n\beta}(\xi)\int_{\Omega^t_{(x,x')}} w^t(\mathrm{d}\xi')\chi_{\Lambda_l,t}(\xi'). \end{aligned}$$

To eliminate the characteristic functions restricting the paths ξ, ξ' in the last integral, we shall use Lemma A.1. First, we estimate the error $\gamma(d)$ when we remove the restriction on the path ξ:

$$\begin{aligned}
\gamma(d) &:= \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x \int_{\Lambda_l} \mathrm{d}x' \frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 t n\beta)^{d/2}} \\
&\quad \times \int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi) e^{-\int_0^{n\beta} \mathrm{d}s\, v^{\omega}(\xi(s))} \big(1 - \chi_{\Lambda_l,n\beta}(\xi)\big) \int_{\Omega^t_{(x,x')}} w^t(\mathrm{d}\xi')\chi_{\Lambda_l,t}(\xi') \\
&\le \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x \int_{\Lambda_l} \mathrm{d}x' \frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 t n\beta)^{d/2}} \int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi)\big(1 - \chi_{\Lambda_l,n\beta}(\xi)\big) \\
&\le \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x \int_{\Lambda_l} \mathrm{d}x' \mathbb{I}\big\{d(x,\partial\Lambda_l) > d(x',\partial\Lambda_l)\big\} \frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 t n\beta)^{d/2}} \\
&\quad \times \int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi)\big(1 - \chi_{\Lambda_l,n\beta}(\xi)\big) \\
&\quad + \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x \int_{\Lambda_l} \mathrm{d}x' \mathbb{I}\big\{d(x,\partial\Lambda_l \le d(x',\partial\Lambda_l)\big\} \frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 t n\beta)^{d/2}} \\
&\quad \times \int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi)\big(1 - \chi_{\Lambda_l,n\beta}(\xi)\big) \\
&\le \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x \int_{\Lambda_l} \mathrm{d}x' K_0^t(x,x') K_0^{n\beta}(x',x) e^{-C(n\beta)(d(x',\partial\Lambda_l)^2} \\
&\quad + \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x \int_{\Lambda_l} \mathrm{d}x' K_0^t(x,x') K_0^{n\beta}(x',x) e^{-C(n\beta)(d(x,\partial\Lambda_l)^2}, \qquad \text{(A.7)}
\end{aligned}$$

where the last step is due to Lemma A.1. Since all integrands are positive, we can extend one of the spatial integrations to the whole space, and hence we get:

$$\begin{aligned}
\gamma(d) &\le \lim_{l\to\infty} \frac{1}{V_l} \int_{\mathbb{R}^d} \mathrm{d}x \int_{\Lambda_l} \mathrm{d}x' K_0^t(x,x') K_0^{n\beta}(x',x) e^{-C(n\beta)(d(x',\partial\Lambda_l)^2} \\
&\quad + \lim_{l\to\infty} \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}x' K_0^t(x,x') K_0^{n\beta}(x',x) e^{-C(n\beta)(d(x'\partial\Lambda_l)^2} \\
&\le \lim_{l\to\infty} \frac{1}{V_l} K_0^{t+n\beta} \int_{\Lambda_l} \mathrm{d}x' e^{-C(n\beta)(d(x',\partial\Lambda_l)^2} + \lim_{l\to\infty} \frac{1}{V_l} K_0^{t+n\beta} \int_{\Lambda_l} \mathrm{d}x e^{-C(n\beta)(d(x'\partial\Lambda_l)^2}
\end{aligned}$$

where we have used the notation $K_0^{t+n\beta} := K_0^{t+n\beta}(x,x)$ since these are independent of x. Finally, using the fact that the boxes Λ_l are cubes of side l, we obtain:

$$\gamma(d) \le \lim_{l\to\infty} \frac{K_0^{t+n\beta}}{l} \int_{-l/2}^{l/2} \mathrm{d}x' e^{-C(n\beta)(l/2-x')^2} + \lim_{l\to\infty} \frac{K_0^{t+n\beta}}{l} \int_{-l/2}^{l/2} \mathrm{d}x e^{-C(n\beta)(l/2-x)^2} = 0.$$

We can estimate the error estimate due to the removal of the characteristic function for ξ' in (4.18) in the same way. Therefore, we get:

$$\begin{aligned}
&\lim_{l\to\infty}\frac{1}{V_l}\int_{\Lambda_l}\int_{\Lambda_l}\mathrm{d}x\mathrm{d}x'\frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}}\\
&\quad\times\int_{\Omega^{n\beta}_{(x',x)}}w^{n\beta}(\mathrm{d}\xi)e^{-\int_0^{n\beta}\mathrm{d}s v^{\omega}(\xi(s))}\chi_{\Lambda_l,n\beta}(\xi)\int_{\Omega^t_{(x,x')}}w^t(\mathrm{d}\xi')\chi_{\Lambda_l,t}(\xi')\\
&=\lim_{l\to\infty}\frac{1}{V_l}\int_{\Lambda_l}\int_{\Lambda_l}\mathrm{d}x\mathrm{d}x'\frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}}\\
&\quad\times\int_{\Omega^{n\beta}_{(x',x)}}w^{n\beta}(\mathrm{d}\xi)e^{-\int_0^{n\beta}\mathrm{d}s v^{\omega}(\xi(s))}\int_{\Omega^p_{(x,x')}}w^{n\beta}(\mathrm{d}\xi').
\end{aligned}\tag{A.8}$$

Now we show that one can replace the first integration over the box Λ_l by one over the whole space. Let $\tilde{\gamma}(d)$ be the error caused by this substitution. Then by the positivity of the random potential we get the estimate:

$$\begin{aligned}
\tilde{\gamma}(d) &:= \lim_{l\to\infty}\frac{1}{V_l}\int_{\mathbb{R}^d\setminus\Lambda_l}\mathrm{d}x\int_{\Lambda_l}\mathrm{d}x'\frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}}\\
&\quad\times\int_{\Omega^{n\beta}_{(x',x)}}w^{n\beta}(\mathrm{d}\xi)e^{-\int_0^{n\beta}\mathrm{d}s v^{\omega}(\xi(s)+x')}\int_{\Omega^t_{(x,x')}}w^{n\beta}(\mathrm{d}\xi')\\
&\le \lim_{l\to\infty}\frac{1}{V_l}\int_{\mathbb{R}^d\setminus\Lambda_l}\mathrm{d}x\int_{\Lambda_l}\mathrm{d}x'\frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}}.
\end{aligned}\tag{A.9}$$

In the one-dimensional case the estimate of the error term (A.9) takes the form:

$$\begin{aligned}
\tilde{\gamma}(1) \le &\lim_{l\to\infty}\frac{1}{l}\int_{-\infty}^{-l/2}\mathrm{d}x\int_{-l/2-x}^{l/2-x}\mathrm{d}y\frac{e^{-y^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{1/2}}\\
&+\lim_{l\to\infty}\frac{1}{l}\int_{l/2}^{\infty}\mathrm{d}x\int_{-l/2-x}^{l/2-x}\mathrm{d}y\frac{e^{-y^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{1/2}}.
\end{aligned}\tag{A.10}$$

For the first term one gets:

$$\begin{aligned}
&\lim_{l\to\infty}\frac{1}{l}\int_{-\infty}^{-l/2}\mathrm{d}x\int_{-l/2-x}^{l/2-x}\mathrm{d}y\frac{e^{-y^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{1/2}}\\
&=\lim_{l\to\infty}\frac{1}{l}\int_0^l\mathrm{d}y\frac{e^{-y^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{1/2}}\int_{-l/2-y}^{l/2}\mathrm{d}x\\
&\quad+\lim_{l\to\infty}\frac{1}{l}\int_l^{\infty}\mathrm{d}y\frac{e^{-y^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{1/2}}\int_{-l/2-y}^{l/2-y}\mathrm{d}x = 0.
\end{aligned}$$

One obtains a similar identity for the second-term in (A.10). Direct calculation shows that, the error term for higher dimensions (A.9) reduces to a product of one-dimensional terms (A.10). Then (A.8) gives:

$$\lim_{l\to\infty}\frac{1}{V_l}\int_{\Lambda_l}\int_{\Lambda_l}\mathrm{d}x\mathrm{d}x'\frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}}$$

$$\times\int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi)e^{-\int_0^{n\beta}\mathrm{d}s v^{\omega}(\xi(s))}\chi_{\Lambda_l,n\beta}(\xi)\int_{\Omega^t_{(x,x')}} w^t(\mathrm{d}\xi')\chi_{\Lambda_l,t}(\xi')$$

$$=\lim_{l\to\infty}\int_{\mathbb{R}^d}\mathrm{d}x\frac{1}{V_l}\int_{\Lambda_l}\mathrm{d}x'\frac{e^{-\|x-x'\|^2(1/2n\beta+1/2t)}}{(4\pi^2 tn\beta)^{d/2}}\int_{\Omega^{n\beta}_{(x',x)}} w^{n\beta}(\mathrm{d}\xi)e^{-\int_0^{n\beta}\mathrm{d}s v^{\omega}(\xi(s))},\quad\text{(A.11)}$$

which finishes the proof of (A.6). □

Appendix B: Some Probabilistic Estimates

First we recall the assumptions on the random potential v^ω used in [14], and which we also adopt in this paper:

1. (a) On the probability space $(\Omega,\mathcal{F},\mathbb{P})$ there exist a group of measure-preserving metrically transitive transformations $\{T_p\}_{p\in\mathbb{R}^d}$ of Ω, such that $v^\omega(x+p)=v^{T_p\omega}(x)$ for all $x,p\in\mathbb{R}^d$;
   (b) $\mathbb{E}_\omega\{\int_{\Lambda_1}\mathrm{d}x|v^\omega(x)|^\kappa\}<\infty$, where $\kappa>\max(2,d/2)$.
2. For any $\Lambda\subset\mathbb{R}^d$, let Σ_Λ be the σ-algebra generated by the random field $v^\omega(x)$, $x\in\Lambda$. For any two arbitrary random variables on Ω, f,g satisfying (i) $|g|_\infty<\infty$, $\mathbb{E}_\omega\{|f|\}<\infty$ and (ii) the function g is Σ_{Λ_1}-measurable, the function f is Σ_{Λ_2}-measurable, where Λ_1,Λ_2 are disjoint bounded subsets of $\mathbb{R}^d$, the following holds

$$|\mathbb{E}\{|f.g|\}-\mathbb{E}\{|f|\}\mathbb{E}\{|g|\}|\le|g|_\infty\mathbb{E}\{|f|\}\phi(d(\Lambda_1,\Lambda_2))$$

with $\phi(x)\to 0$ as $x\to\infty$, and $d(\Lambda_1,\Lambda_2)$ the Euclidean distance between Λ_1 and Λ_2.

After recalling these conditions, we can give a sketch of the proof of Lemma 5.1.

Let $h_l^{\omega,N}$ to be the Schrödinger operator (2.3), with Neumann boundary conditions instead of Dirichlet, and denote by $\{E_i^{\omega,l,N},\phi_i^{\omega,l,N}\}_{i\ge1}$ its ordered eigenvalues (including degeneracy) and the corresponding eigenvectors. Similarly we define the kinetic energy operator $h_l^{0,N}$ with the same boundary condition, and denote by $\{\varepsilon_k^{l,N},\psi_k^{l,N}\}_{k\ge1}$ its ordered eigenvalues (including degeneracy) and corresponding eigenvectors. The following result is due to Thirring, see [28]:

Lemma B.1 *Let* $v^\omega_{\lambda,\alpha}:=v^\omega+\lambda\alpha$, *for* $\lambda,\alpha>0$. *Then,*

$$E_1^{\omega,l,N}\ge-\lambda\alpha+\min\left\{\varepsilon_2^{l,N},\left[\frac{1}{V_l}\int_{\Lambda_l}\mathrm{d}x\,(v^\omega_{\lambda,\alpha}(x))^{-1}\right]^{-1}\right\}.$$

Proof Let P to be an orthogonal projection in $\mathscr{H}_l$. Then for any vector ϕ from the intersection $Q(v^\omega_{\lambda,\alpha})\cap Q((v^\omega_{\lambda,\alpha})^{1/2}P(v^\omega_{\lambda,\alpha})^{1/2})$, we have:

$$\begin{aligned}(\phi,v^\omega_{\lambda,\alpha}\phi)&=((v^\omega_{\lambda,\alpha})^{1/2}\phi,(v^\omega_{\lambda,\alpha})^{1/2}\phi)\\&=((v^\omega_{\lambda,\alpha})^{1/2}\phi,P(v^\omega_{\lambda,\alpha})^{1/2}\phi)+((v^\omega_{\lambda,\alpha})^{1/2}\phi,(1-P)(v^\omega_{\lambda,\alpha})^{1/2}\phi)\\&\ge((v^\omega_{\lambda,\alpha})^{1/2}\phi,P(v^\omega_{\lambda,\alpha})^{1/2}\phi),\end{aligned}$$

and therefore,

$$-\frac{1}{2}\Delta_N + v^{\omega}_{\lambda,\alpha} \geq -\frac{1}{2}\Delta_N + (v^{\omega}_{\lambda,\alpha})^{1/2} P (v^{\omega}_{\lambda,\alpha})^{1/2}, \tag{B.1}$$

in the quadratic-form sense. Let us choose:

$$P := (v^{\omega}_{\lambda,\alpha})^{-1/2} \tilde{P} \big((\psi_1^{l,N}, (v^{\omega}_{\lambda,\alpha})^{-1} \psi_1^{l,N}) \big)^{-1} \tilde{P} (v^{\omega}_{\lambda,\alpha})^{-1/2},$$

where $\tilde{P}$ is the orthogonal projection onto the subspace spanned by the vector $\psi_1^{l,N}$. It can be easily checked that P is an orthogonal projection. Applying (B.1) to the function $\phi_1^{\omega,l,N}$ one gets:

$$\begin{aligned} E_1^{\omega,l,N} + \lambda\alpha &\geq \left(\phi_1^{\omega,l,N}, \left(-\frac{1}{2}\Delta_N \right) \phi_1^{\omega,l,N} \right) + |(\phi_1^{\omega,l,N}, \psi_1^{l,N})|^2 \big(\psi_1^{l,N}, (v^{\omega}_{\lambda,\alpha})^{-1} \psi_1^{l,N} \big)^{-1} \\ &\geq \sum_{k\geq 1} |(\phi_1^{\omega,l,N}, \psi_k^{l,N})|^2 \varepsilon_k^{l,N} + |(\phi_1^{\omega,l,N}, \psi_1^{l,N})|^2 \left[\frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x (v^{\omega}_{\lambda,\alpha}(x))^{-1} \right]^{-1}. \end{aligned}$$

But since the Neumann boundary conditions imply that $\varepsilon_1^{l,N} = 0$, we obtain

$$E_1^{\omega,l,N} + \lambda\alpha \geq (1 - |(\phi_1^{\omega,l,N}, \psi_1^{l,N})|^2) \varepsilon_2^{l,N} + |(\phi_1^{\omega,l,N}, \psi_1^{l,N})|^2 \left[\frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x (v^{\omega}_{\lambda,\alpha}(x))^{-1} \right]^{-1}.$$

To finish the proof, we have to study separately the two cases, namely, $\varepsilon_2^{l,N}$ less than and greater than $[\frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x (v^{\omega}_{\lambda,\alpha}(x))^{-1}]^{-1}$. □

Proof of Lemma 5.1 By Lemma B.1,with $\lambda = B/l^2$ and α as defined in assumptions, i.e. for $B = \pi/(1+\alpha)$, $\alpha > p/(1-p)$, we have:

$$E_1^{\omega,l,N} \geq -\frac{\alpha B}{l^2} + \min(\pi/l^2, 1/X_l),$$

where

$$X_l^{\omega} := \frac{1}{V_l} \int_{\Lambda_l} \mathrm{d}x \frac{1}{v^{\omega}(x) + B\alpha/l^2}.$$

Therefore,

$$E_1^{\omega,l,N} - \frac{B}{l^2} \geq -\frac{\pi}{l^2} + \min(\pi/l^2, 1/X_l^{\omega}).$$

Hence, the inequality $E_1^{\omega,l,N} < B/l^2$ implies that $X_l^{\omega} > l^2/\pi$ and consequently:

$$\mathbb{P}(E_1^{\omega,l,N} < B/l^2) \leq \mathbb{P}(X_l^{\omega} > l^2/\pi). \tag{B.2}$$

Define a random variable $Y_l^{\omega}(\delta) := V_l^{-1} \int_{\Lambda_l} \mathrm{d}x\ \delta/(v^{\omega}(x) + \delta)$, which is an increasing function of δ. Then for the left-hand side of (B.2) one gets the estimate:

$$\mathbb{P}(E_1^{\omega,l,N} < B/l^2) \leq \mathbb{P}\left(Y_l^{\omega}(B\alpha/l^2) > \frac{\alpha}{1+\alpha} \right).$$

By Lemma 2 in [14], we know that for any positive δ the random variables $\{Y_l^\omega(\delta)\}_l$, converges *geometrically* to a limit $Y_\infty(\delta)$ as $l \to \infty$, that is, for any $\epsilon > 0$, there exists a constant $M(\delta,\epsilon)$ such that

$$\mathbb{P}(|Y_l^\omega(\delta) - Y_\infty(\delta)| > \epsilon/2) \leq e^{-M(\delta,\epsilon)V_l}$$

for l sufficiently large. By the ergodic theorem $Y_\infty(\delta)$ is non-random and can be expressed as:

$$Y_\infty(\delta) = \mathbb{E}_\omega\left(\frac{\delta}{v^\omega(0)+\delta}\right),$$

which is again a monotonic function of $\delta \geq 0$. Notice that by condition (ii), Sect. 2, we have $\lim_{\delta\to 0} Y_\infty(\delta) = p$.

Choose $\epsilon > 0$ such that $p + \epsilon < \alpha/(1+\alpha)$. Then we have

$$\mathbb{P}\left(E_1^{\omega,l,N} < \frac{B}{l^2}\right) \leq \mathbb{P}\big(Y_l^\omega(B\alpha/l^2) > p + \epsilon\big).$$

Now we choose δ such that

$$Y_\infty(\delta) - p < \epsilon/2,$$

and let l_0 be defined by $\delta = B\alpha/l_0^2$. Then for any $l > l_0$ we have:

$$\begin{aligned}\mathbb{P}(E_1^{\omega,l,N} < B/l^2) &\leq \mathbb{P}\big(Y_l^\omega(B\alpha/l^2) > p+\epsilon\big) \leq \mathbb{P}\big(Y_l^\omega(\delta) - p > \epsilon\big) \\ &\leq \mathbb{P}\big(|Y_l^\omega(\delta) - Y_\infty(\delta)| > \epsilon/2\big) \leq e^{-M(\delta,\epsilon)V_l}. \qquad \square\end{aligned}$$

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